Information Quality, Disagreement and Political Polarisation

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Abstract

How does the quality of information received by voters affect political polarisation? We address this long-standing question using an election competition model in which voters have to infer an unknown state from some noisy and biased signals. Their policy preferences are shaped by the posterior belief, which is unknown to the parties when they choose their platforms. The greater the uncertainty faced by the parties, the greater the incentive to polarise. We show that better information can either promote or suppress polarisation, depending on the gap between voters' and politicians' beliefs (disagreement). We also examine the welfare implications of polarisation.

Keywords: Polarisation, Voter Information, Bayesian Learning, Election

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1 Introduction

A well-informed electorate is crucial for the functioning of representative democracy. Knowledge and information about the state of the world, among other things, provide the basis for voters to form their opinions and elect the representatives that best promote their interests. But in reality, voters (and politicians alike) often have to make electoral decisions before conclusive evidence or complete knowledge is available, for example, when deciding on policies that have long-term social and economic repercussions, or when drafting measures to handle an unprecedented global pandemic. The absence of an evidence-based consensus means that voters are susceptible to conflicting news and biased information. It is well-documented that major elections in recent years have been plagued by misinformation among voters and escalating polarisation between political parties [see, for instance, Grinberg *et al.* (2019), Chen *et al.* (2021), and Munger *et al.* (2022)]. What is less explored is the potential mechanism that links the two. The present study is intended to address this important question. Specifically, we examine the theoretical linkages between voters' political information processing and belief formation on the one side, and political parties' strategic policy choices on the other. Our results highlight the importance of two factors which are often overlooked in the existing literature, namely (i) perceived biasedness of the information sources and (ii) disagreement between voters' and politicians' beliefs. The role and significance of these features will become clear in the following paragraphs.

Our analysis is based on a prototypical two-party electoral competition model in which voters' policy preferences are contingent on an unknown state of the world.[1] Thus, similar to the model of Calvert (1985), Roemer (1994) and Bernhardt *et al.* (2009), the median voter's ideal policy is not *a priori* known to the political parties. In the presence of such uncertainty, the political gain from choosing a moderate platform is likely to be small. This induces ideologically differentiated parties to abandon the middle ground and choose policies that are closer to their own ideals. Bernhardt *et al.* (2009) show that such an environment can pave the way for policy polarisation in equilibrium. In particular, political parties have a stronger incentive to polarise if they are more uncertain about the electorate's preferences. In their framework, voters' preferences are largely driven by the exogenous state. Thus, polarisation depends on the parties' perceived uncertainty about the hidden state, but not on the voters' belief. We depart from their study by assuming

[1] This can be interpreted either as some unanticipated major events (e.g., economic crisis, foreign wars, pandemic) that can sway public opinion in an election, or as the optimal policy response to an issue that has far-reaching consequences (e.g., immigration, abortion rights, economic reforms). Throughout this paper, we will use the terms "hidden state", "unknown state", and "policy issue" interchangeably.

that voters are Bayesian learners, who infer the unknown state from the news and information that they receive before the election. As is standard in Bayesian learning models, the learners' posterior belief is determined by two factors. The first one is their subjective prior belief, which captures their pre-existing worldview. In our model, this includes any information and knowledge that the voters possess *before* the political parties announce their platforms. The second factor is a set of publicly observed signals. These represent the dissemination of political news and information from both formal channels (such as mainstream news media, official government announcements, political endorsements, etc.) and informal ones (such as social media) *after* the parties announce their platforms.[2] We enrich the standard learning model by assuming that the random signals are not only fraught with potential errors, they may also be biased. This added feature is motivated by the extensive empirical evidence on the pervasiveness of biased reporting in mass media.[3] In our model, the biasedness of each signal is captured by an additive random bias term. Voters possess prior belief about the hidden state and the biasedness of each information channel, but they cannot separately identify these factors from the observed signals.[4]

The electorate's policy preferences are shaped by their posterior belief about the hidden state. This introduces an explicit channel through which voters' information processing and belief formation can affect their electoral decisions. But, importantly, this posterior belief is unknown to the politicians when they choose their platforms because the signals are realised afterwards.[5] Thus, in the decision stage political parties form expectation about the voters' posterior belief based on (i) their perceived uncertainty of the signals and (ii) how voters will respond to those signals.[6] The former depends on the parties' assessment of the hidden state and the quality of the signals (i.e., their precision and biasedness). Greater uncertainty means that the parties are less able to correctly predict the information received by the voters, and hence their posterior belief and policy

[2] The timing of the signals makes clear that we are focusing on the voters' learning process after the parties announced their platforms but before the election.

Conceptually, these signals may include rumors (i.e., statements that are not backed by sufficient evidence) and disinformation (i.e., false information which is intended to mislead, such as propaganda). The presumption here is that voters are unable to distinguish these from true information. This opens up the possibility for rumors and disinformation to affect voters' belief formation.

[3] See Puglisi and Snyder (2015) for a comprehensive survey on the empirical evidence of biased reporting in traditional news media (newspapers and cable news). A more recent study by Garz *et al.* (2020) provide evidence on political media slant during the 2012 and 2016 US presidential elections. For an in-depth discussion about the political effects of social media, see Zhuravskaya *et al.* (2020).

[4] A similar learning model with biased signals is also considered in Little and Pepinsky (2021) and Little *et al.* (2022).

[5] This setup captures the following ideas: Before entering the election booth, voters are free to adjust their policy stance upon the arrival of any new information. But political parties are less likely to make significant changes in their platforms before the election to avoid any potential damages on credibility and reputation.

[6] We assume that the political parties have perfect knowledge on *how* the voters will update their belief under a given set of signals. Thus, the random signals are the ultimate source of uncerainty faced by the parties.

preferences. This will incentivise the parties to polarise. We refer to this as the *uncertainty effect*. Voters' responsiveness to the signals, on the other hand, depends on their subjective assessment on the hidden state and signal quality (which may not coincide with the parties' assessment). For instance, if the voters are unfamiliar with the policy implications of the hidden state (as captured by a low precision of their prior belief), or if they believe the signals are of good quality, then they will rely more heavily on the signals in the learning process. From the parties' perspective, this means the voters are more easily influenced by the random signals and hence their policy preferences are less predictable. This will again encourage the parties to polarise. We refer to this as the *learning effect*. Using this framework, we examine how the precision and the perceived biasedness of the signals will affect the extent of policy polarisation in equilibrium. Our main findings are summarised as follows.

Our first set of results concerns the effects of an improvement in signal precision. We assume that the statistical properties of the signal errors are common knowledge among voters and politicians. Thus, any changes in signal precision will affect both sides. On the one hand, more precise signals will boost the voters' confidence on the learning process and promote polarisation through the learning effect. But on the other hand, when the signals become less noisy, politicians can better predict the information received by the voters, which weaken the uncertainty effect and lower polarisation. Which effect dominates depends crucially on the voters' and parties' subjective prior beliefs about the hidden state. In the current study, we abstract away from belief heterogeneity among voters and between the two parties.[7] Instead, we focus on the disagreement in beliefs between voters and political candidates. We first explain the implications of this model feature, followed by a brief discussion on its empirical relevance. If the two sides share the same prior belief about the hidden state (as is commonly assumed in the existing literature), then the learning effect dominates so that more precise signals will promote policy polarisation. The learning effect will continue to dominate if the voters hold a stronger (or more precise) prior belief about the hidden state than the politicians. But if the politicians are more confident about their prior estimate, then the uncertainty effect will dominate and better signal precision will lower polarisation. As an illustration, consider the extreme case in which voters have a "flat prior," i.e., the precision of their prior belief is zero. Then their posterior belief will simplify mirror the distribution of the random signals (or a sufficient statistic of the signals).[8] In this case, the learning effect described above is

[7] Further discussions about this and other major assumptions can be found at the end of Section 2.

[8] For instance, if the signals are unbiased, normally distributed and independent of each other (Case 1 in Section 3), then voters will use the average value of the realised signals as their updated estimate of the hidden state.

not operative and any improvement in signal precision will reduce polarisation through the uncertainty effect. If we interpret the unknown state as the optimal policy response to a particular issue, then our first set of results yield the following implications: If voters share a strong pre-existing view on how to best handle the issue, then more precise signals will promote polarisation. But for issues that they know less about, then an improvement in signal precision will lower polarisation.

Three additional remarks are in order. First, our first set of results are robust under different specifications of the signal process. In particular, these results are valid under both biased and unbiased signals, and when the signals are correlated. Second, for correlated signals, we show that reducing the correlation coefficient between the signals will have the same effect as improving their precision. Intuitively, voters will have more confidence in the learning process if they perceive the news that they consumed as independent opinions rather than "echo chambers". This will strengthen the learning effect and encourage policy polarisation. Reducing signal correlation, however, will also reduce the uncertainty faced by the parties. This will weaken the uncertainty effect and reduce polarisation. The net effect again depends on the two sides' prior beliefs as described above. Third, the above discussion highlights the importance of disagreement between voters and politicians in characterising our results. This raises a natural question of whether this type of disagreement is empirically relevant. The answer is positive: There is ample evidence showing that political elites and their staff often misperceive their constituent's opinions and preferences [see, for instance, Broochman and Skovron (2018), Hertel-Fernandez *et al.* (2019), Pereira (2021), Kärnä and Öhberg (2023)]. Several hypotheses have been put forward and investigated by the existing studies, including (1) politicians' worldview and opinions are shaped by their own socioeconomic and educational background, which may differ from their constituents, (2) elected officials' opinions are more influenced by a subset of their constituency, such as activists, interest groups, lobbyists and businesses [Giger and Klűver (2016)], and (3) elected officials often disregard opinions and views that disagree with their own [Butler and Dynes (2016)]. In the present study, we do not take a stance on why such disagreement exists. Instead, we focus on its implications on political polarisation.[9]

Our second set of results concerns the perceived biasedness of the signals. We assume that both voters and politicians expect the signals to be unbiased in their prior beliefs.[10] The variance

[9] Kärnä and Öhberg (2023) provide an interesting and insightful account on how the disagreement between the elected officials and the electorate in Sweden on immigration policies may have contributed to the rise of populist parties and political polarisation.

[10] This assumption does not affect the voters' learning process because they will subtract the prior mean of the hidden state and the bias terms from the observed signals when forming their posterior estimate. See Lemma 1 for a formal statement and proof of this result.

of these beliefs then capture their confidence on the impartiality of the news sources. A lower variance (or higher precision) indicates that they are more confident in this regard. As before, we do not require the two sides to share a common prior about the bias terms. We first consider the case in which the bias terms are statistically independent of the hidden state.[11] Introducing this type of biases will simply add more noises to the signals. Thus, similar to an improvement in signal precision, when voters become more confident about the news that they consumed, the learning effect will be intensified which in turn promote polarisation. On the other hand, if politicians become more confident about the news sources, then polarisation will be less likely and less severe due to a weakened uncertainty effect.[12]

The analysis becomes much more complicated when the bias terms are either positively or negatively correlated with the unknown state. In order to simplify the analysis, we focus our attention to only one random signal in this part. The signal is called exaggerating [resp., contradicting] if the bias term is positively [resp., negatively] correlated with the hidden state.[13] Regarding voters' learning, one interesting implication of a contradicting signal is that voters may engage in what we call "signal-defiant" learning, i.e., they update their belief in the *opposite* direction as suggested by the signal.[14] This draws some similarities with COVID-19 deniers (or conspiracy theorists in general) who distrust the mainstream news medias and government officials, and often misinterpret or distort the information provided by these sources. This type of learning is not possible under the conventional Bayesian model with unbiased signals. But regardless of which direction they update their belief, polarisation will become more likely and more severe when voters are more responsive to the signals. We provide a thorough analysis on how and when this will happen under three scenarios: (i) when voters become more certain or more knowledgeable about the hidden state in their prior belief, (ii) when voters become more confident about the impartiality of the signal, and (iii) when the biased term is more correlated with the hidden state. Even with this relatively simple framework, there is a great variety of possible cases and non-monotonic relations. It is difficult to explain these results clearly and precisely without first introducing some technical

[11] In all the cases that we considered, the bias terms are independent of the signal errors and across each other.

[12] Since voters and politicians can have different subjective prior beliefs about signal biases, these two effects can happen independently.

[13] Examples of contradicting signals include rumors and disinformation that discredit the scientific evidence behind man-made climate changes or the efficacy of vaccination.

[14] For example, suppose the observed signal (m) is favourable to the right-wing candidate, i.e., $m > 0$. If the voters believe that the signal contains a bias term (b) that is strongly negatively correlated with the actual state (s), then they may interpret $m > 0$ as the result of a strongly positive bias $(b > 0)$ when the actual state is negative $(s < 0)$. This type of reasoning will direct them to update their beliefs in the opposite direction as suggested by the observed signal. In our model, signal-defiant learning happens only when s and b are sufficiently negatively correlated so that s and m are negatively correlated.

details. For this reason, we defer an in-depth discussion to Section 3 Case 4. As for the politicians, we show that greater confidence in the impartiality of the signal will lower polarisation even if the signal is exaggerating or mildly contradicting.

Finally, our third set of results concerns how changes in policy polarisation will affect the *ex ante* welfare (i.e., before the signals are realised) of an arbitrary voter. From a risk-averse voter's perspective, policy divergence can provide a partial insurance against the uncertainty in their ideal policy [Bernhardt *et al.* (2009, p.573)] and thus can be welfare-improving.[15] Too much polarisation, however, will turn a boon into a bane. There is thus an acceptable range of polarisation within which all risk-averse voters will strictly prefer policy divergence to convergence, and beyond which they are strictly worse off under policy divergence. The size of this range is determined by the voters' perceived uncertainty about their ideal policy. Greater uncertainty will create a higher demand for the insurance provided by polarisation which then widens the acceptable range. The extent of polarisation in equilibrium, on the other hand, is determined by the parties' choices and their subjective belief. In particular, higher uncertainty shared by the parties will encourage them to polarise. This shows that voters' belief and politicians' belief each plays a different role in shaping the welfare results.

Against this backdrop, we find that when there is little disagreement between voters and politicians, voters strictly prefer a society with partisan parties and *any* positive level of polarisation to an otherwise identical society but with more congruent parties and convergent platforms. It seems surprising that this is true even for highly polarised policy platforms. Intuitively, equilibrium policies are very polarised only when the parties perceive a high level of uncertainty on voters' ideal policies. This high level of uncertainty is shared by the voters when the prior beliefs of the two groups coincide. Consequently, voters would prefer polarised policies. However, when there is strong disagreement between the two, this result will no longer hold. In other words, parties can be polarised to such an extent that voters would rather have convergent platforms.

An improvement in signal quality can potentially lead to a welfare loss, but only when there is significant disagreement between voters' and politicians' beliefs.[16] This can happen in either one of the following two ways: (1) Better signal quality lowers the parties' perceived uncertainty and induces them to narrow the gap between their platforms. This in turn reduces the insurance

[15] In equilibrium, the outcome of the election is determined by the median voter's ideal policy, which is ultimately determined by the realised signals. Thus, for an arbitrary voter, the perceived uncertainty about the election outcome is shaped by her prior belief about the signals (or more precisely, the hidden state, the bias terms and the signal errors combined).

[16] Ashworth and Bueno de Mesquita (2014) also present different setups where better voter information can be welfare-reducing due to the strategic interaction between voters and politicians.

provided by polarisation. (2) Better signal quality increases the parties' perceived uncertainty by strengthening the learning effect. Polarisation increases significantly as a result and goes beyond the voters' acceptable range, leading to a welfare loss. We provide two sets of numerical examples to demonstrate that both scenarios are possible.

Related Literature The present study contributes to the growing literature that examines the effect of voter information on policy outcomes. Each of these studies discussed below, however, focuses on a different mechanism from the one that we considered. Gul and Pesendorfer (2012) consider how the media industry structure and voter polarisation can affect the candidate endorsement strategies of the profit-maximising media and consequently policy choices of parties. When the number of media firms approaches infinity, each voter is able to find a media firm that endorses her favorite party in each state of the world. This means that the electorate behaves as if perfectly informed and this leads to policy polarisation. In Levy and Razin (2015), voters receive private signals about the state of the world, and if voters have correlation neglect, they become more sensitive to the signals and therefore their beliefs become more dispersed. Levy and Razin (2015) show that this does not necessarily lead to policy polarisation. They also note that correlation neglect of voters makes the information aggregation more efficient, which increases voter welfare. In Yuksel (2022), voters differ in the policy dimensions they find important, and when they specialise in their learning accordingly, this leads to more dispersed voter beliefs and consequently to more policy polarisation. However, for a given level of specialised learning, better access to information of voters leads to reduced party polarisation. In Yuksel (2022)'s model, policy polarisation always reduces voter welfare. In a related paper to Yuksel (2022), Perego and Yuksel (2022) show how media competition leads to informational specialization across voters and consequently to social disagreement. Personalised demand for information leads to inefficient policy outcomes in Matejka and Tabellini (2021). Similarly, personalised news aggregators result in different types of voters (centrist and extreme voters) receiving different information and potentially lead to policy polarisation in Lin et al. (2023).

On a different vein, the political agency literature studies the implications of voter information on electoral accountability and selection. When the incumbent politician type's is private information, better voter information creates a trade-off between accountability and selection in Besley and Smart (2007) and Smart and Sturm (2013). Ashworth et al. (2017) show that such a trade-off exists even with symmetric information when the politician effort and type are complementary in the production function of public goods. Li and Lin (2023) study the effect of personalised news aggregators on electoral accountability and selection.

Finally, in the empirical literature on political polarisation, there is a consensus that party polarisation in the U.S. is on the rise (McCarty *et al.,* 2006), but whether voters are now more polarised is less clear (Barber and McCarty, 2015). The explanation we provide for party polarisation does not rely on voter polarisation, but rather on the interaction between voter information and the disagreement between voters and politicians.

The rest of the paper is organised as follows. Section 2 presents the model environment. Section 3 presents the main results under four different specifications of the signal process. Section 4 examines the welfare implications of policy polarisation. Section 5 concludes.

2 The Model

Consider an election in which two political parties, L and R, compete on a one-dimensional policy issue. Prior to the election, the two parties simultaneously propose a policy from the policy space $X \equiv \mathbb{R}$. The electorate consists of a continuum of voters with heterogeneous policy preferences. The size of the electorate is normalised to one. Each voter v's policy preferences are determined by two factors: (i) a deterministic parameter $\delta_v \in \mathbb{R}$ which captures the voter's pre-existing political attitudes, and (ii) a random variable $s \in \mathbb{R}$ which captures the exogenous state of the world. In any given state s, voter v's utility from policy $x \in \mathbb{R}$ is given by

$$U\left(x;\delta_v\right) = -\left(\delta_v + s - x\right)^2.$$

The median of δ_v across voters is normalised to zero. The exact distribution of δ_v is irrelevant to our analysis.

Voters do not observe the realisation of s at the time of the election.[17] Instead, they receive imperfect information about s from $n \geq 1$ sources. Each information channel $i \in \{1, 2, ..., n\}$ produces a noisy signal m_i which is potentially biased. Let $m_i = b_i + s + \varepsilon_i$, where b_i is the bias and ε_i is the error term, for all $i \in \{1, 2, ..., n\}$. Voters share the same subjective prior belief about the state variable s and the biases $\mathbf{b} = (b_1, ..., b_n)^T$.[18] This is assumed to take the form of a joint

[17] Based on our interpretations of s in Footnote 1, this means the full effect of the unforeseen major event, or the optimal policy response to a certain issue, is unknown when the election takes place.

[18] We will discuss the rationale behind this and other major assumptions at the end of this section.

multivariate normal distribution $\mathbf{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, where

$$\boldsymbol{\mu}_0 = \begin{bmatrix} \mu_s \\ \boldsymbol{\mu}_b \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\Sigma}_0 = \begin{bmatrix} \sigma_s^2 & \boldsymbol{\Omega}^T \\ \boldsymbol{\Omega} & \boldsymbol{\Sigma}_b \end{bmatrix}.$$

In the above expressions, μ_s and σ_s^2 are scalars representing the mean and variance of the marginal distribution of s; whereas $\boldsymbol{\mu}_b$ and $\boldsymbol{\Sigma}_b$ are the mean vector and the covariance matrix of the marginal distribution of $\mathbf{b}$.[19] The covariances between s and $\mathbf{b}$ are captured by the 1-by-n row vector $\boldsymbol{\Omega}^T = (\omega_1, ..., \omega_n)$, where $\omega_i \equiv Cov(s, b_i)$. A positive value of ω_i means that the bias term b_i tends to exaggerate or complement the effect of the hidden state variable, whereas a negative value means that b_i tends to contradict the effect of s. Voters' subjective prior belief may not coincide with the true distribution of $(s, \mathbf{b})$ and it may also differ from the political parties' belief about the same variables.

The error terms, $\boldsymbol{\varepsilon} = (\varepsilon_1, ..., \varepsilon_n)^T$, are drawn from a normal distribution $\mathbf{N}(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$. Each ε_i is independent of the distribution of δ_v and the voters' prior belief about $(s, \mathbf{b})$. The statistical properties of $\boldsymbol{\varepsilon}$ are known to both voters and political parties. In other words, there is no disagreement regarding the distribution of the error terms.

Given the voters' prior belief, the signals $\mathbf{m}$ have a joint normal distribution with mean vector

$$\boldsymbol{\mu}_m = \mu_s \cdot \mathbf{1}_n + \boldsymbol{\mu}_b,$$

where $\mathbf{1}_n$ is an n-by-1 column of ones, and covariance matrix

$$\begin{aligned} \boldsymbol{\Sigma}_m &= E\left[(\mathbf{m} - \boldsymbol{\mu}_m)(\mathbf{m} - \boldsymbol{\mu}_m)^T\right] \\ &= \boldsymbol{\Sigma}_b + \sigma_s^2 \cdot \mathbf{1}_n \mathbf{1}_n^T + \boldsymbol{\Sigma}_\varepsilon + \boldsymbol{\Omega} \mathbf{1}_n^T + \boldsymbol{\Omega}^T \mathbf{1}_n. \end{aligned} \tag{1}$$

Equation (1) suggests that the quality of the signals (as measured by the inverse of $\boldsymbol{\Sigma}_m$) is determined by three groups of factors:[20] (i) the precision of the voters' subjective prior belief, as captured by the inverse of $\boldsymbol{\Sigma}_b$ and the scalar value $\tau_s \equiv \sigma_s^{-2}$, (ii) the precision of the signal errors, as captured by the inverse of $\boldsymbol{\Sigma}_\varepsilon$, and (iii) the covariances between s and $\mathbf{b}$, which are contained in $\boldsymbol{\Omega}$.

[19] All the covariance matrices appeared in this study are assumed to be (at least) positive semidefinite.

[20] Except for some special cases (such as those considered in Section 3), there is no general formula for $\boldsymbol{\Sigma}_m^{-1}$. Hence, the discussion here should be considered as heuristic in nature.

Before the election, voters observe the same set of signals $\mathbf{m} = (m_1, ..., m_n)^T$ but not the realised values of s or $\mathbf{b}$. They then update their belief about $(s, \mathbf{b})$ using Bayes' rule. The resulting posterior belief is again a multivariate normal distribution. For the purpose of our analysis, it suffice to focus on the marginal distribution of s in the posterior beliefs which is characterised in Lemma 1.[21] In order to state this result, we need to introduce two additional notations: Define $\mathbf{\Lambda} \equiv E\left[(s-\mu_s)(\mathbf{m}-\boldsymbol{\mu}_m)^T\right]$, which is a 1-by-$n$ row vector capturing the covariance between s and $\mathbf{m}$. The ith element of $\mathbf{\Lambda}$ is $\lambda_i \equiv Cov(s, m_i) = \sigma_s^2 + \omega_i$. Let $\kappa_{i,j}$ be the element on the ith row and jth column of the precision matrix $\mathbf{\Sigma}_m^{-1}$.

Lemma 1 *The marginal distribution of s in the voters' posterior belief is a normal distribution with mean*

$$E(s \mid \mathbf{m}) = \mu_s + \sum_{j=1}^{n} \alpha_j \left(m_j - \mu_s - \mu_{b_j}\right), \tag{2}$$

and variance

$$var(s \mid \mathbf{m}) = \sigma_s^2 - \sum_{j=1}^{n} \lambda_j \alpha_j, \tag{3}$$

where $\alpha_j \equiv \sum_{i=1}^{n} \lambda_i \kappa_{i,j}$ for all $j \in \{1, 2, ..., n\}$.

Unless otherwise stated, all proofs can be found in the Appendix. As an illustrative example, consider the case when there is only one biased signal, i.e., $n = 1$. The covariance matrix of the voters' prior belief can be simplified to become

$$\mathbf{\Sigma}_0 = \begin{bmatrix} \sigma_s^2 & \rho_{s,b}\sigma_s\sigma_b \\ \rho_{s,b}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix},$$

where σ_b^2 is the variance of the bias b and $\rho_{s,b} \in (-1, 1)$ is the correlation coefficient between s and b. The matrices $\mathbf{\Lambda}$ and $\mathbf{\Sigma}_m$ are now replaced by the scalars $\lambda = Cov(s, m) = \sigma_s^2 + \rho_{s,b}\sigma_s\sigma_b$ and

$$var(m) = \sigma_s^2 + 2\rho_{s,b}\sigma_s\sigma_b + \sigma_b^2 + \sigma_\varepsilon^2,$$

respectively. Note that $Cov(s, m) \gtrless 0$ if and only if $\rho_{s,b} \gtrless -\sigma_s/\sigma_b$. Thus, a negative correlation between s and b is necessary but not sufficient for $Cov(s, m) < 0$. In other words, a mildly contradicting bias term (i.e., $-\sigma_s/\sigma_b < \rho_{s,b} < 0$) can still generate a positive covariance between

[21] The full details of the posterior distribution of $(s, \mathbf{b})$ are shown in the proof of Lemma 1 located in the Appendix.

s and m. The expressions in (2) and (3) can now be simplified to become[22]

$$E(s \mid m) = \mu_s + \frac{Cov(s,m)}{var(m)}(m - \mu_s - \mu_b), \tag{4}$$

$$var(s \mid m) = \sigma_s^2 - \frac{[Cov(s,m)]^2}{var(m)}. \tag{5}$$

Equation (4) shows that only the difference $(m - \mu_s - \mu_b)$ matters when forming the posterior expectation $E(s \mid m)$. In particular, voters will adjust the observed signal either upward or downward according to the prior means (μ_s, μ_b). Equation (5) shows that learning can always reduce voters' uncertainty about s, i.e., $var(s \mid m) < \sigma_s^2$, even when the signal is perceived to be biased and when it is negatively correlated to the hidden state. The size of the reduction (i.e., the gain from learning) is negatively related to σ_ε^2, which means more can be learned from a more precise signal. The effect of changing σ_b^2 on $var(s \mid m)$, however, depends on parameter values. We will examine this and other special cases more fully in Section 3.

Given the posterior belief about s, voter v's expected utility from policy x is given by

$$E[U(x;\delta_v) \mid \mathbf{m}] = E\left[-(\delta_v + s - x)^2 \mid \mathbf{m}\right]. \tag{6}$$

The voter's ideal policy, δ_v^*, is one that maximises (6), i.e.,

$$\begin{aligned} \delta_v^* &\equiv \arg\max_{x \in \mathbb{R}} \left\{E\left[-(\delta_v + s - x)^2 \mid \mathbf{m}\right]\right\} \\ &= \delta_v + E(s \mid \mathbf{m}). \end{aligned} \tag{7}$$

Equations (2) and (7) together show how voters use the observed signals to form their policy preferences. Let $\{x_R, x_L\}$ be the policies proposed by the two parties. If $x_R = x_L$, then voters are indifferent between the two. If $x_R \neq x_L$, then after observing $\mathbf{m}$, voter v will choose x_R over x_L if and only if

$$\begin{gathered} -(\delta_v^* - x_R)^2 > -(\delta_v^* - x_L)^2 \\ \Leftrightarrow (x_R - x_L)(\delta_v^* - \overline{x}) > 0, \end{gathered} \tag{8}$$

where $\overline{x} = (x_L + x_R)/2$. Hence, voter v will support R if either (i) $x_R > x_L$ and $\delta_v^* > \overline{x}$, or (ii) $x_R < x_L$ and $\delta_v^* < \overline{x}$. The voter is indifferent between any $x_R \neq x_L$ if $\overline{x} = \delta_v^*$.

[22] The same equation for $E(s \mid m)$ also appears in Little and Pepinsky (2021, p.610) but in a very different context. Their study is not directly related to electoral competition and policy polarisation.

The two political parties are assumed to be both office-motivated and policy-motivated. This means they not only care about their chance of winning, but also the policy implemented by the winner of the election. The parties' preferences on policy x are represented by

$$U(\phi_k, x) = -(x - \phi_k)^2,$$

where $\phi_k \in \mathbb{R}$ is the ideal policy of party $k \in \{L, R\}$. If R wins, then x_R is implemented and it receives a payoff of $-(x_R - \phi_R)^2 + \gamma$, where $\gamma \geq 0$ represents the additional benefits of holding office. If R loses, then its payoff is $-(x_L - \phi_R)^2$. The payoffs for L are defined symmetrically.

Events in the model unfold in three stages: First, the two parties simultaneously choose a policy that maximises their own expected utility. Both parties are fully aware of the median value of δ_v, the probability distribution of $\boldsymbol{\varepsilon}$ and the voters' prior belief about $(s, \mathbf{b})$. Hence, the parties are also fully aware of the updating rule in (2). The two parties, however, do not observe the realisation of $\mathbf{m}$ and s when they make their choices. Hence, they will act according to their expectations on $E(s \mid \mathbf{m})$. In the second stage, the signals $\mathbf{m}$ are realised and made public. Voters then update their belief according to (2) and (3), and choose a party based on (8). Following Bernhardt *et al.* (2009), it is assumed that the political parties cannot revoke or adjust their policy platforms at this stage. Finally, the party that garners a majority of votes wins.

We now focus on the first stage of events and characterise the parties' policy choices. We begin by formulating the parties' winning probability. If $x_L = x_R$, then the winner is decided by a fair coin toss. Suppose $x_L \neq x_R$. Given $\mathbf{m}$ and (8), R wins if it gains the median voter's support. Since the median value of δ_v is normalised to zero, the median voter's ideal policy is captured by $E(s \mid \mathbf{m})$ alone. Hence, R wins if either (i) $x_R > x_L$ and $E(s \mid \mathbf{m}) > \overline{x}$, or (ii) $x_R < x_L$ and $E(s \mid \mathbf{m}) < \overline{x}$. The value of $E(s \mid \mathbf{m})$, however, is unknown to the parties as $\mathbf{m}$ is not yet revealed at this stage. The parties' *perceived* probability of winning thus depends on their *perceived* probability distribution of $\mathbf{m}$, which in turn hinges on their subjective beliefs about $(s, \mathbf{b})$. In order to capture the separate effects of voters' belief and parties' belief on policy polarisation, we depart from the existing literature by allowing them to be different.

More specifically, we assume the two political parties share a common belief about $(s, \mathbf{b})$, which is given by a normal distribution $\mathbf{N}\left(\widehat{\boldsymbol{\mu}}_0, \widehat{\boldsymbol{\Sigma}}_0\right)$ with

$$\widehat{\boldsymbol{\mu}}_0 = \begin{bmatrix} \widehat{\mu}_s \\ \widehat{\boldsymbol{\mu}}_b \end{bmatrix} \qquad \text{and} \qquad \widehat{\boldsymbol{\Sigma}}_0 = \begin{bmatrix} \widehat{\sigma}_s^2 & \widehat{\boldsymbol{\Omega}}^T \\ \widehat{\boldsymbol{\Omega}} & \widehat{\boldsymbol{\Sigma}}_b \end{bmatrix}.$$

The elements of $\widehat{\boldsymbol{\mu}}_0$ and $\widehat{\boldsymbol{\Sigma}}_0$ can be interpreted similarly as those of $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$. Under this belief, each signal m_i has an expected value $E_p(m_i) = \widehat{\mu}_s + \widehat{\mu}_{b_i}$. The covariance structure among the n signals is determined by

$$Cov_p(m_i, m_j) = Cov_p(b_i, b_j) + \widehat{\sigma}_s^2 + Cov_p(s, b_i) + Cov_p(s, b_j),$$

where $Cov_p(b_i, b_j)$ is the (i,j)th element of $\widehat{\boldsymbol{\Sigma}}_b$ and $Cov_p(s, b_i)$ is the ith element of $\widehat{\boldsymbol{\Omega}}$, for all i, $j \in \{1, 2, ..., n\}$. We use the subscript "p" to indicate that these moments are derived from the parties' belief. It follows that, from the parties' perspective, $E(s \mid \mathbf{m})$ is a normal random variable with mean

$$E_p[E(s \mid \mathbf{m})] \equiv \widetilde{\mu} = \mu_s + \sum_{j=1}^{n} \alpha_j \left[(\widehat{\mu}_s - \mu_s) + \left(\widehat{\mu}_{b_j} - \mu_{b_j}\right)\right] \tag{9}$$

and variance

$$var_p\{E(s \mid \mathbf{m})\} \equiv \widetilde{\sigma}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j Cov_p(m_i, m_j). \tag{10}$$

A higher value of $\widetilde{\sigma}^2$ means that the parties are more uncertain about the median voter's policy ideal. This variance is in the centre stage of our analysis, and we will refer to it as the parties' *perceived uncertainty.* Equation (10) shows that this variable not only depends on the parties' subjective belief, but also on the voters' prior belief and the precision of the signals which are embedded in $\{\alpha_1, ..., \alpha_n\}$.

Let $H(\cdot)$ be the cumulative distribution function of $N\left(\widetilde{\mu}, \widetilde{\sigma}^2\right)$, and $h(\cdot)$ be the corresponding probability density function. Then R's winning probability is given by

$$\Pr[E(s \mid \mathbf{m}) > \overline{x}] = \begin{cases} 1/2 & \text{if } x_R = x_L, \\ 1 - H(\overline{x}) & \text{if } x_R > x_L, \\ H(\overline{x}) & \text{if } x_R < x_L. \end{cases} \tag{11}$$

Notice that apart from $x_R = x_L$, the two parties have equal opportunity of winning if $\overline{x}$ coincides with the median of the distribution $N\left(\widetilde{\mu}, \widetilde{\sigma}^2\right)$, which is $\widetilde{\mu}$. We will refer to this as the centrist policy position. Party R is deemed as the "right-wing" party if its ideal policy ϕ_R is on the right side of the centrist position, i.e., $\phi_R > \widetilde{\mu}$. Similarly, party L is the left-wing party if $\phi_L < \widetilde{\mu}$. Since the actual value of $\widetilde{\mu}$ is immaterial to the following analysis, it is normalised to zero from this point onward. This is achieved by setting $\mu_s = 0$ in the voters' prior belief and having $\boldsymbol{\mu}_0 \equiv \widehat{\boldsymbol{\mu}}_0$ so that voters' and parties' beliefs differ only in the covariance matrices. We further assume that the two parties' ideal policies are symmetric, i.e., they are equidistant on both sides of $\widetilde{\mu} = 0$, so that $\phi_R = -\phi_L = \phi > 0$.

Taking $x_L \in \mathbb{R}$ as given, party R's policy choice problem is to choose $x_R \in \mathbb{R}$ so as to maximise its expected utility

$$\mathcal{W}_R(x_R; x_L) = \left[-(x_R - \phi)^2 + \gamma\right] \Pr\left[E(s \mid \mathbf{m}) > \overline{x}\right] - (x_L - \phi)^2 \left\{1 - \Pr\left[E(s \mid \mathbf{m}) > \overline{x}\right]\right\},$$

subject to (11). Let $\mathcal{B}_R(x_L)$ denote R's best-response correspondence under a given $x_L \in \mathbb{R}$, i.e.,

$$\mathcal{B}_R(x_L) \equiv \arg\max_{x_R \in \mathbb{R}} \left\{\mathcal{W}_R(x_R; x_L)\right\}.$$

Party L's expected utility $\mathcal{W}_L(x_L; x_R)$ and best-response correspondence $\mathcal{B}_L(x_R)$ are similarly defined.

We focus on pure-strategy Nash equilibria of the voting game. Specifically, a voting equilibrium is a pair of policies $(x_R^*, x_L^*) \in \mathbb{R}^2$ such that $x_R^* \in \mathcal{B}_R(x_L^*)$ and $x_L^* \in \mathcal{B}_L(x_R^*)$. It can be shown that any voting equilibrium, if exists, must satisfy[23]

$$-\phi < x_L^* \leq x_R^* < \phi. \tag{12}$$

The main intuition is straightforward: Given that $\phi_R > \phi_L$, it is never optimal for R to choose a policy to the left of x_L^*, and likewise it is never optimal for L to choose a policy to the right of x_R^*. Hence, $x_R^* < x_L^*$ cannot occur in any voting equilibrium. Similar to Bernhardt *et al.* (2009), we further confine our attention to *symmetric* equilibrium, i.e., one in which x_R^* and x_L^* are equidistant on both sides of the centrist position, so that $x_R^* = -x_L^* = x_{eq}^* \geq 0$. Policy convergence is said to occur if $x_{eq}^* = 0$. Policy divergence or polarisation, on the other hand, refers to $x_R^* \neq x_L^*$. In a symmetric equilibrium, this happens when $x_{eq}^* > 0$.

The following result, which is due to Bernhardt *et al.* (2009, Corollary 2), provides a detailed

[23] This result is well-known in the existing literature and is often stated without proof. A detailed proof of this statement can be found on the authors' personal website.

characterisation of symmetric equilibrium.[24] The model in Bernhardt *et al.* (2009), however, differs from ours in one important regard: In their framework, voters observe the realised value of s before the election and $\widetilde{\sigma}$ is an exogenous parameter. In the present study, voters have imperfect information about s and $\widetilde{\sigma}$ is endogenously determined by their learning process and politicians' belief. This allows us to examine how the quality of information received and possessed by the two groups will affect political polarisation.

Proposition 1 *(a) If $\phi \leq \gamma h(0)/2$, then there exists a unique symmetric equilibrium in which both parties choose the same policy which is the centrist position, i.e., $x_{eq}^* = 0$.*

(b) If $\phi > \gamma h(0)/2$, then there exists a unique symmetric equilibrium in which the two parties choose different policies, i.e., $x_R^* = -x_L^* = x_{eq}^* > 0$ and x_{eq}^* is given by

$$x_{eq}^* = \frac{2\phi - \gamma h(0)}{4h(0)\phi + 2}. \tag{13}$$

This result can be explained by considering the first-order condition of party R's policy choice problem (conditional on $x_R \geq x_L$), which is

$$\frac{\partial \mathcal{W}_R(x_R; x_L)}{\partial x_R} = -2(x_R - \phi)[1 - H(\overline{x})] + \frac{1}{2}\left[(x_R - \phi)^2 - (x_L - \phi)^2 - \gamma\right] h(\overline{x}) \leq 0. \tag{14}$$

Taking $x_L \in \mathbb{R}$ as given, suppose R is thinking about moving its policy from some $x_R \geq x_L$ to $x_R + \Delta$, where $\Delta > 0$ is infinitesimal. Conditional on winning, such a move will bring R closer to its ideal policy and raise its utility by $2(x_R - \phi)\Delta$. This happens with probability $[1 - H(\overline{x})]$. Hence, the expected marginal benefit from this is $2(x_R - \phi)[1 - H(\overline{x})]\Delta$. The same move, however, will take R further away from its opponent's policy and jeopardise its winning opportunity. The associated loss in expected utility is given by

$$\left[(x_R - \phi)^2 - (x_L - \phi)^2 - \gamma\right] h(\overline{x})\Delta/2.$$

When evaluated at $x_R = x_L = 0$, the condition in (14) can be simplified to become

$$\left.\frac{\partial \mathcal{W}_R(x_R; 0)}{\partial x_R}\right|_{x_R=0} = \phi - \frac{1}{2}\gamma h(0) \leq 0.$$

[24] Bernhardt *et al.* (2009, Proposition 4) establish the same result without imposing the assumptions of quadratic utility and normal distribution. For the sake of completeness and consistency in notations, we provide a detailed proof of Proposition 1 in an (unpublished) online technical appendix available from the author's website.

If $\phi \leq \gamma h(0)/2$, then the expected marginal cost of moving from $x_R = 0$ to $x_R = \Delta$ is greater than the expected marginal benefit. Hence, it is optimal for R to choose $x_R = x_L = 0$.[25] If instead $\phi > \gamma h(0)/2$, then the first-order condition in (14) will have a unique interior solution. The first-order condition of party L's policy choice problem can be interpreted in the same fashion.

The main message of Proposition 1 is that the additional benefits of holding office γ (which captures the strength of the parties' office motivation) must be sufficiently large in order to induce the parties to sacrifice their own political ideals (policy motivation) and move towards their opponent's policy position. Since $h(0) \equiv 1/\left(\widetilde{\sigma}\sqrt{2\pi}\right)$ for the normal distribution $N\left(0, \widetilde{\sigma}^2\right)$,

$$\phi \lesseqgtr \gamma h(0)/2 \qquad \text{if and only if} \qquad \widetilde{\sigma} \lesseqgtr \sigma_{\min} \equiv \gamma/(2\sqrt{2\pi}\phi).$$

This states that $\sigma_{\min}$ is a unique threshold value of $\widetilde{\sigma}$ below which office motivation will dominate policy motivation so that policy convergence will emerge, and above which policy motivation will dominate and policy polarisation will occur. Put it differently, heightened uncertainty shared by the political parties will weaken their office motivation. Hence, policy polarisation is more likely to happen and more severe when $\widetilde{\sigma}$ increases. This observation is summarised in Corollary 1, which follows immediately by differentiating (13) with respect to $\widetilde{\sigma}$.

Corollary 1 *Assume $\phi > \gamma h(0)/2$. Then, the degree of policy polarisation is strictly increasing in $\widetilde{\sigma}$, i.e.,*

$$\frac{dx^*_{eq}}{d\widetilde{\sigma}} = \frac{h(0)}{\widetilde{\sigma}} \frac{2\left(\gamma + 4\phi^2\right)}{\left[4h(0)\phi + 2\right]^2} > 0. \qquad (15)$$

Remarks on Model Assumptions Before proceeding further, we first discuss several key assumptions in our model. The first one is the assumption that all voters share the same prior belief about $(s, \mathbf{b})$ and receive the same set of signals. This is mainly for the sake of simplifying the analysis. Note that if we allow for heterogeneity in both δ_v and the learning process (e.g., heterogeneous priors among voters or privately observed signals), then there will not be a single decisive voter in the model. Instead, the election outcome will be decided by those voters whose ideal policy after observing the signals (δ^*_v) is at the median position across voters, i.e., any v that

[25] The condition $\phi \leq \gamma h(0)/2$ also implies that the objective function $\mathcal{W}_R(x_R; 0)$ is strictly decreasing in x_R for all $x_R \geq 0$. Thus, $x_R = 0$ is R's unique best response to $x_L = 0$.

satisfies

$$\delta_{med}^{*}=\delta_{v}+E_{v}\left(s\mid\mathbf{m}_{v}\right),$$

where E_v is the expectation operator based on voter v's posterior belief after observing $\mathbf{m}_v$ (which will differ across voters if we allow for private signals). This will greatly increase the complexity of the analysis and also make it much harder to interpret the results. Hence, we do not follow this route.

The second major assumption in our model is that the two political parties share a common belief about $(s, \mathbf{b})$. This is mainly chosen to suit the purpose of studying *symmetric* equilibrium. In Bernhardt *et al.* (2009), the two political parties are assumed to have the same utility function, receive the same benefits from holding office and share the same belief about the hidden state. The only difference between them is their policy ideals, which are equally distanced on both sides of the centrist position.[26] These assumptions set the stage for defining and characterising symmetric voting equilibrium. Our common prior assumption between the two parties can be viewed as a natural extension of this tradition.

Finally, we assume that there is no disagreement between voters and politicians regarding signal precision (i.e., $\mathbf{\Sigma}_\varepsilon$). Our analysis in the following sections can be easily extended to accommodate this type of disagreement. In all the cases that we considered, we report the separate effect of signal precision on the voters' learning process and the politicians' perceived uncertainty about the signals.

3 Special Cases

In this section we examine how the quality of information possessed by voters and political parties will affect the extent of policy polarisation in equilibrium. In particular, we focus on five aspects of information and beliefs, namely (i) the precision of the voters' and political parties' prior beliefs about s and $\{b_1, ..., b_n\}$; (ii) the precision of the errors in the signals; (iii) the disagreement between voters' and parties' prior beliefs as captured by $\mathbf{\Sigma}_0$ and $\widehat{\mathbf{\Sigma}}_0$; (iv) the pairwise correlation between different signals; and (v) the perceived correlation between the state variable and the biases.

In order to convey the main results in a clear and parsimonious manner, we focus on a series of special cases. In each of these cases, voters' posterior expectation of s and parties' perceived

[26] The same assumptions on political parties are also adopted by Ossokina and Swank (2004), Saporiti (2008), Xefteris and Zudenkova (2018), among many others.

uncertainty can be expressed as

$$E(s \mid \mathbf{m}) = \psi \widehat{m} \qquad \text{and} \qquad \widetilde{\sigma}^2 = \psi^2 var_p(\widehat{m}),$$

where $\widehat{m}$ is a sufficient statistic of the observed signals $\{m_1, ..., m_n\}$ and $\psi > 0$ captures the responsiveness of voters' posterior expectation to $\widehat{m}$. The exact form of ψ and $\widehat{m}$ vary across cases, but several general principles apply to all. First, ψ is determined by the voters' learning process. It thus depends on the quality of information available to the voters and their subjective prior belief (i.e., $\mathbf{\Sigma}_\varepsilon$ and $\mathbf{\Sigma}_0$), but is independent of the parties' beliefs. An increase in ψ will encourage polarisation by raising the parties' perceived uncertainty $\left(\widetilde{\sigma}^2\right)$. We refer to this mechanism as the *learning effect.* Second, $var_p(\widehat{m})$ summarises the parties' subjective uncertainty regarding $\widehat{m}$. It thus depends solely on the information available to the parties and their subjective belief (i.e., $\mathbf{\Sigma}_\varepsilon$ and $\widehat{\mathbf{\Sigma}}_0$), but not on the voters'. An increase in $var_p(\widehat{m})$ will increase the uncertainty faced by the parties and promote polarisation. We refer to this as the *uncertainty effect.* Third, any changes in the precision of the error terms $\{\varepsilon_1, ..., \varepsilon_n\}$ will have opposite effects on ψ and $var_p(\widehat{m})$. The overall effect on $\widetilde{\sigma}^2$ depends on the relative magnitude between $2var_p(\widehat{m})$ and $var(\widehat{m})$, where $var(\widehat{m})$ is the unconditional variance of $\widehat{m}$ under the voters' subjective prior belief. The details of these points will be explained more fully in each of the special cases.

Case 1: Unbiased Independent Signals

We begin with the case in which (i) both voters and politicians believe with certainty that all n signals are unbiased so that each b_i is a deterministic constant and normalised to zero, and (ii) the error terms $\{\varepsilon_1, ..., \varepsilon_n\}$ are independently drawn from different probability distributions. Specifically, each ε_i is assumed to be drawn from a normal distribution $N\left(0, \tau_{\varepsilon_i}^{-1}\right)$, where τ_{ε_i} is the precision of m_i. The expressions of $E(s \mid \mathbf{m})$, $var(s \mid \mathbf{m})$ and $\widetilde{\sigma}^2$ are shown in Lemma 2.[27]

Lemma 2 *Suppose all the signals are unbiased and each ε_i is independently drawn from the distribution $N\left(0, \tau_{\varepsilon_i}^{-1}\right)$ for all i. Define ψ and $\widehat{m}$ according to*

$$\psi \equiv \frac{\sum_{i=1}^n \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}} > 0 \qquad \text{and} \qquad \widehat{m} \equiv \sum_{i=1}^n \zeta_i m_i, \tag{16}$$

[27] The proof of Lemmas 2 and 3 also serve as a demonstration on how to apply the formulas in (2) and (3). We are aware of other (simpler) methods that can derive the posterior mean and posterior variance when signals are unbiased.

where $\tau_s \equiv \sigma_s^{-2}$ *and* $\zeta_i \equiv \tau_{\varepsilon i} / \sum_{i=1}^{n} \tau_{\varepsilon i}$ *for all* i*. Then the mean and variance of* s *in the voters' posterior beliefs are given by*

$$E(s \mid \mathbf{m}) = \psi \widehat{m} \qquad \textit{and} \qquad var(s \mid \mathbf{m}) = \frac{1}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon i}}. \tag{17}$$

The political parties' perceived uncertainty is given by $\widetilde{\sigma}^2 = \psi^2 var_p(\widehat{m})$*, where*

$$var_p(\widehat{m}) \equiv \frac{\left(\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon i}\right)}{\widehat{\tau}_s \left(\sum_{i=1}^{n} \tau_{\varepsilon i}\right)}, \tag{18}$$

and $\widehat{\tau}_s \equiv \widehat{\sigma}_s^{-2}$.

In this special case, the summary measure $\widehat{m}$ is a weighted average of all the signals whereby more precise signals are weighted more heavily. If the error terms $\{\varepsilon_1, ..., \varepsilon_n\}$ are i.i.d. normal random variables, so that $\tau_{\varepsilon_i} = \tau_\varepsilon$ for all i, then the summation $\sum_{i=1}^{n} \tau_{\varepsilon i}$ in (16)-(18) will be replaced by $n\tau_\varepsilon$. The parties' perceived uncertainty then becomes

$$\widetilde{\sigma}^2 = \frac{n\tau_\varepsilon \left(\widehat{\tau}_s + n\tau_\varepsilon\right)}{\widehat{\tau}_s \left(\tau_s + n\tau_\varepsilon\right)^2}.$$

On the other hand, if voters and parties share the same subjective prior beliefs about s so that $\widehat{\tau}_s = \tau_s$, then the parties' perceived uncertainty becomes

$$\widetilde{\sigma}^2 = \frac{\left(\sum_{i=1}^{n} \tau_{\varepsilon i}\right)}{\tau_s \left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon i}\right)}.$$

Recall that policy polarisation will emerge in a symmetric equilibrium if and only if $\widetilde{\sigma}^2$ exceeds a certain threshold value $\sigma_{\min}^2$. Thus, understanding the relations between $\{\tau_s, \widehat{\tau}_s, \tau_{\varepsilon_1}, ..., \tau_{\varepsilon_n}\}$ and $\widetilde{\sigma}^2$ is essential in understanding how quality of information and disagreement will affect policy polarisation.[28] To this end, we first examine the effects of changing $\{\tau_s, \widehat{\tau}_s, \tau_{\varepsilon_1}, ..., \tau_{\varepsilon_n}\}$ on $\widetilde{\sigma}^2$ in Proposition 2.

Proposition 2 *Suppose all the signals are unbiased and each* ε_i *is independently drawn from the distribution* $N\left(0, \tau_{\varepsilon_i}^{-1}\right)$ *for all* i*.*

(a) *Holding other factors constant, an increase in either* τ_s *or* $\widehat{\tau}_s$ *will lower the value of* $\widetilde{\sigma}^2$*.*

(b) *Holding other factors constant, an increase in* τ_{ε_i}*, for any* $i \in \{1, 2, ..., n\}$*, will raise the*

[28] The threshold value $\sigma_{\min}^2$ itself is independent of the precision parameters $\{\tau_s, \widehat{\tau}_s, \tau_{\varepsilon_1}, ..., \tau_{\varepsilon_n}\}$.

value of ψ but lower the value of $var_p(\widehat{m})$.

(c) *Holding other factors constant,*

$$\frac{d\widetilde{\sigma}^2}{d\tau_{\varepsilon_i}} \gtrless 0 \qquad \text{if and only if} \qquad 2var_p(\widehat{m}) \gtrless var(\widehat{m}), \tag{19}$$

for any $i \in \{1, 2, ..., n\}$.

The first part of Proposition 2 states that policy polarisation is less likely to emerge and less severe when either voters or parties are more certain about the hidden state in their prior beliefs. This result can be easily explained through the learning effect and the uncertainty effect. As voters become more certain about s, they will be less reliant on the signals in the learning process. Consequently, their posterior expectation will be less responsive to $\widehat{m}$ (i.e., ψ decreases). From the parties' perspective, this means less *ex ante* uncertainty in the median voter's ideal policy $E(s \mid \mathbf{m})$, hence a lower value of $\widetilde{\sigma}^2$.[29] As explained before, this will strengthen the parties' office motivation and incentivise them to move closer to their opponent's position in order to boost their winning probability. Hence, an increase in τ_s will lower polarisation by weakening the learning effect. An increase in $\widehat{\tau}_s$, on the other hand, has no impact on the voters' learning process. But as the parties' become more certain about the hidden state, they also perceive the signals as less uncertain. This suppresses the uncertainty effect and reduces the extent of polarisation.

The other parts of Proposition 2 analyse the effects of changing a single τ_{ε_i} on $\widetilde{\sigma}^2$. Part (b) shows that such a change will have opposite effects on ψ and $var_p(\widehat{m})$. Firstly, having more precise signals will encourage voters to become more reliant on them when updating their beliefs. This will enhance polarisation by strengthening the learning effect. An increase in τ_{ε_i} also means that the signal m_i becomes more precise which will curb the uncertainty effect and lower polarisation. To determine the overall effect on $\widetilde{\sigma}^2$, consider the following decomposition of $\ln \widetilde{\sigma}^2$,

$$\frac{d \ln \widetilde{\sigma}^2}{d \ln \tau_{\varepsilon_i}} = 2\frac{d \ln \psi}{d \ln \tau_{\varepsilon_i}} + \frac{d \ln var_p(\widehat{m})}{d \ln \tau_{\varepsilon_i}}. \tag{20}$$

The first term on the right captures the changes in $\widetilde{\sigma}^2$ due to the learning effect, while the second term captures the contribution of the uncertainty effect. As shown in the proof of Proposition 2,

[29] In the extreme case when τ_s is arbitrarily large, $var(s \mid \mathbf{m})$ will converge to zero and $E(s \mid \mathbf{m})$ will converge to the expected value of s in the prior distribution, which is $\mu_s = 0$. The median voter's ideal policy then converges to the median value of δ_v, which is a known constant. This eliminates the uncertainty faced by the parties and paves the way for policy convergence.

the contribution of the learning effect is inversely related to $var(\widehat{m})$. Specifically,

$$\frac{d\ln\psi}{d\ln\tau_{\varepsilon_i}} = \frac{\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n}\tau_{\varepsilon_i}\right)^2}\frac{1}{var(\widehat{m})} > 0. \tag{21}$$

The intuition of this is as follows: First, note that

$$var(\widehat{m}) = \tau_{\varepsilon_i}^{-1} + \sum_{j\neq i}\tau_{\varepsilon_j}^{-1} + \tau_s^{-1}.$$

If voters are highly uncertain about $\widehat{m}$ to begin with [e.g., due to a low value of τ_s or τ_{ε_j}, for $j \neq i$], then a one-percentage increase in τ_{ε_i} will have a small impact on $var(\widehat{m})$ and the outcome of the learning process. On the same vein, the contribution of the uncertainty effect is inversely related to $var_p(\widehat{m})$, i.e.,

$$\frac{d\ln var_p(\widehat{m})}{d\ln\tau_{\varepsilon_i}} = -\frac{\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n}\tau_{\varepsilon_i}\right)^2}\frac{1}{var_p(\widehat{m})} < 0. \tag{22}$$

By combining (20)-(22), we can show that which effect dominates depends on the relative magnitude between $2var_p(\widehat{m})$ and $var(\widehat{m})$.

We can also express this condition in terms of the precision parameters. In the current special case, $2var_p(\widehat{m}) \gtrless var(\widehat{m})$ if and only if

$$\tau_s \gtrless \frac{\widehat{\tau}_s\sum_{i=1}^{n}\tau_{\varepsilon_i}}{\widehat{\tau}_s + 2\sum_{i=1}^{n}\tau_{\varepsilon_i}}. \tag{23}$$

Thus, improving the precision of the noisy signals will increase [resp., reduce] perceived uncertainty and polarisation if and only if τ_s is greater [resp., less] than a threshold that is determined by $\widehat{\tau}_s$ and $\sum_{i=1}^{n}\tau_{\varepsilon_i}$. Notice that if there is no disagreement between voters and parties so that $\tau_s = \widehat{\tau}_s$ and $var(\widehat{m}) = var_p(\widehat{m})$, then more precise signals will *always* lead to an increase in $\widetilde{\sigma}^2$ and polarisation.[30] This is no longer the case when voters and political parties disagree. In particular, if the political parties are sufficiently more certain or more knowledgeable on the policy issue (the hidden state) so that $2var_p(\widehat{m}) < var(\widehat{m})$, then more precise signal(s) will reduce polarisation.[31]

[30] A similar result is reported in Gul and Pesendorfer (2012, Lemma 2). The main focus of Gul and Pesendorfer (2012), however, is on the relation between media competition and party polarisation.

[31] Note that the expression on the right side of (23) is strictly lower than $\widehat{\tau}_s$. Hence, part (c) of Proposition 2 implies

$$\frac{d\widetilde{\sigma}^2}{d\tau_{\varepsilon_i}} < 0 \quad \text{iff} \quad \tau_s < \frac{\widehat{\tau}_s\sum_{i=1}^{n}\tau_{\varepsilon_i}}{\widehat{\tau}_s + 2\sum_{i=1}^{n}\tau_{\varepsilon_i}} < \widehat{\tau}_s.$$

Case 2: Unbiased, Correlated and Exchangeable Signals

In this subsection we maintain the assumption that all signals are (believed to be) unbiased so that $b_i = 0$ for all i, but the error terms $\{\varepsilon_1, ..., \varepsilon_n\}$ are now assumed to be exchangeable normal random variables. Specifically, this means each ε_i has the same marginal distribution with mean zero and precision τ_ε, and each pair $(\varepsilon_i, \varepsilon_j)$, $i \neq j$, has the same covariance. The covariance matrix $\mathbf{\Sigma}_\varepsilon$ is now given by

$$\mathbf{\Sigma}_\varepsilon = \frac{1}{\tau_\varepsilon} \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & & \ddots & \vdots \\ \rho & \cdots & \rho & 1 \end{bmatrix}, \tag{24}$$

where $\rho \geq -1/(n-1)$ is the correlation coefficient between any pair $(\varepsilon_i, \varepsilon_j)$, $i \neq j$. The lower bound of ρ is necessary for $\mathbf{\Sigma}_\varepsilon$ to be positive semi-definite. The resulting expressions of $E(s \mid \mathbf{m})$, $var(s \mid \mathbf{m})$ and $\widetilde{\sigma}^2$ are shown in Lemma 3.

Lemma 3 *Suppose all the signals are unbiased and the error terms $\{\varepsilon_1, ..., \varepsilon_n\}$ are exchangeable normal random variables with zero mean vector and covariance matrix $\mathbf{\Sigma}_\varepsilon$ as shown in (24). Define* ψ and $\widehat{m}$ according to

$$\psi \equiv \frac{n\tau_\varepsilon}{n\tau_\varepsilon + \tau_s [1 + (n-1)\rho]} > 0 \qquad \textit{and} \qquad \widehat{m} \equiv \frac{1}{n}\sum_{i=1}^{n} m_i.$$

Then the mean and variance of s in the voters' posterior beliefs are given by

$$E(s \mid \mathbf{m}) = \psi \widehat{m} \qquad \textit{and} \qquad var(s \mid \mathbf{m}) = \frac{1 + (n-1)\rho}{n\tau_\varepsilon + \tau_s [1 + (n-1)\rho]}.$$

The political parties' perceived uncertainty is given by $\widetilde{\sigma}^2 = \psi^2 var_p(\widehat{m})$, where

$$var_p(\widehat{m}) = \frac{n\tau_\varepsilon + \widehat{\tau}_s [1 + (n-1)\rho]}{n\tau_\varepsilon \widehat{\tau}_s}.$$

The results in Proposition 2 can be readily extended to the current case with only minor changes. These are formally stated in the first three parts of Proposition 3. The interpretations are essentially the same as before, hence they are not repeated here.

Proposition 3 *Suppose all the signals are unbiased and the error terms $\{\varepsilon_1, ..., \varepsilon_n\}$ are exchangeable normal random variables with zero mean vector and covariance matrix $\mathbf{\Sigma}_\varepsilon$ as shown in (24).*

(a) *Holding other factors constant, an increase in either τ_s or $\widehat{\tau}_s$ will lower the value of $\widetilde{\sigma}^2$.*

(b) *Holding other factors constant, an increase in τ_ε will raise the value of ψ but lower the value of $var_p(\widehat{m})$.*

(c) *Holding other factors constant,*

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} \gtrless 0 \qquad \text{if and only if} \qquad 2var_p(\widehat{m}) \gtrless var(\widehat{m}). \tag{25}$$

(d) *Holding other factors constant, an increase in ρ will lower the value of ψ but raise the value of $var_p(\widehat{m})$.*

(e) *Holding other factors constant,*

$$\frac{d\widetilde{\sigma}^2}{d\rho} \gtrless 0 \qquad \text{if and only if} \qquad 2var_p(\widehat{m}) \lesseqgtr var(\widehat{m}). \tag{26}$$

The last two parts of Proposition 3 concern the effects of ρ on $\widetilde{\sigma}^2$. A higher value of ρ means that the signals $\{m_1, ..., m_n\}$ are more correlated. In the extreme case when $\rho = 1$, all the signals are essentially echoing each other. From the voters' perspective, observing $n > 1$ perfectly correlated signals is no better than observing a single one in terms of learning the hidden state s. Thus, a more positive value of ρ will erode the voters' confidence on the signals and weaken the learning effect.[32] The same increase in ρ, however, also raises the parties' perceived variance of $\widehat{m}$, strengthening the uncertainty effect. The overall effect on $\widetilde{\sigma}^2$ again depends on the relative magnitude between $2var_p(\widehat{m})$ and $var(\widehat{m})$. Interestingly, the condition in (26) is the exact opposite of the one in (25). This means, for any given set of $\{\tau_s, \widehat{\tau}_s, \tau_\varepsilon, n, \rho\}$, τ_ε and ρ tend to have opposite effects on $\widetilde{\sigma}^2$.

In the current special case, $2var_p(\widehat{m}) \gtrless var(\widehat{m})$ if and only if

$$\tau_s \gtrless \frac{n\tau_\varepsilon\widehat{\tau}_s}{2n\tau_\varepsilon + \widehat{\tau}_s[1 + (n-1)\rho]}.$$

Similar to Case 1, if there is no disagreement between voters' and politicians' beliefs so that $var_p(\widehat{m}) = var(\widehat{m})$, then an increase in the precision of the signals or a decrease in the correlation

[32] The same idea has been put forward by Ortoleva and Snowberg (2015, p.518), but they have not explored the relation between perceived sigal correlation and policy polarisation.

between signals will raise the parties' perceived uncertainty. However, when voters and politicians disagree, it is possible that an increase in τ_ε or a decrease in ρ will lead to a lower degree of perceived uncertainty. As in Case 1, this happens when $\widehat{\tau}_s$ is sufficiently higher than τ_s or when ρ is sufficiently low.

Case 3: Biased Signals (I)

Suppose now both voters and political parties expect the signals to be unbiased in their prior beliefs but they are not entirely sure about this. Specifically, both groups share the same belief that (i) each b_i is drawn from a normal distribution with mean zero, (ii) the bias terms $\{b_1, ..., b_n\}$ are mutually independent, so that $Cov(b_i, b_j) = Cov_p(b_i, b_j) = 0$, for all $i \neq j$, and (iii) each b_i is independent of the state variable s so that $Cov(b_i, s) = Cov_p(b_i, s) = 0$. Voters and politicians, however, may have different degrees of confidence on the unbiasedness of the signal sources. Let τ_{b_i} and $\widehat{\tau}_{b_i}$ denote, respectively, the precision of b_i in the voters' and the parties' prior beliefs. A higher value of τ_{b_i} [resp., $\widehat{\tau}_{b_i}$] means that voters [resp., politicians] become more firmly believed in the impartiality of the source of m_i. As in Case 1, it is assumed that each ε_i is independently drawn from the distribution $N\left(0, \tau_{\varepsilon_i}^{-1}\right)$. The current special case can thus be viewed as a generalisation of Case 1 with the addition of independent random biases.

Lemma 4 *Suppose both voters and parties believe that each ε_i is independently drawn from $N\left(0, \tau_{\varepsilon_i}^{-1}\right)$ for all i, but they disagree on the precision of the hidden state and the bias terms, i.e., $\tau_s \neq \widehat{\tau}_s$ and $\tau_{b_i} \neq \widehat{\tau}_{b_i}$. Define ψ and $\widehat{m}$ according to*

$$\psi \equiv \frac{\sum_{i=1}^n \widetilde{\tau}_i}{\tau_s + \sum_{i=1}^n \widetilde{\tau}_i} > 0 \qquad \text{and} \qquad \widehat{m} \equiv \sum_{i=1}^n \widetilde{\zeta}_i m_i, \tag{27}$$

where $\widetilde{\tau}_i \equiv \left(\tau_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1}\right)^{-1}$ and $\widetilde{\zeta}_i \equiv \widetilde{\tau}_i / \sum_{i=1}^n \widetilde{\tau}_i$ for all i. Then the mean and variance of s in the voters' posterior beliefs are given by

$$E(s \mid \mathbf{m}) = \psi \widehat{m} \qquad \textit{and} \qquad var(s \mid \mathbf{m}) = \frac{1}{\tau_s + \sum_{i=1}^n \widetilde{\tau}_i}.$$

The political parties' perceived uncertainty is given by $\widetilde{\sigma}^2 = \psi^2 var_p(\widehat{m})$, where

$$var_p(\widehat{m}) = \frac{\left(\sum_{i=1}^n \widetilde{\tau}_i\right)^2 + \widehat{\tau}_s \sum_{i=1}^n \widetilde{\tau}_i^2 \left(\widehat{\tau}_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1}\right)}{\widehat{\tau}_s \left(\sum_{i=1}^n \widetilde{\tau}_i\right)^2}.$$

When comparing the expressions in (27) to those in (16), it is clear that we are now replacing $\{\tau_{\varepsilon_1}, ..., \tau_{\varepsilon_n}\}$ in the latter with $\{\widetilde{\tau}_1, ..., \widetilde{\tau}_n\}$, where each $\widetilde{\tau}_i \equiv \left(\tau_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1}\right)^{-1}$. Intuitively, adding a set of independent bias terms is similar to adding more noises to the signals. In particular, an increase in either τ_{b_i} or τ_{ε_i} will raise the value of $\widetilde{\tau}_i$, so that

$$\frac{d\widetilde{\tau}_i}{d\tau_{b_i}} > 0 \qquad \text{and} \qquad \frac{d\widetilde{\tau}_i}{d\tau_{\varepsilon_i}} > 0 \quad \text{for all } i.$$

But the nonlinearity of $\widetilde{\tau}_i = \left(\tau_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1}\right)^{-1}$ also creates a complementarity between τ_{b_i} and τ_{ε_i}, i.e.,

$$\frac{d^2\widetilde{\tau}_i}{d\tau_{\varepsilon_i} d\tau_{b_i}} > 0.$$

Similar to part (a) of Propositions 2 and 3, an increase in either τ_s or $\widehat{\tau}_s$ will lower the value of $\widetilde{\sigma}^2$. The intuition is the same as before. Here we focus on the effects of changing $\{\tau_{b_i}, \widehat{\tau}_{b_i}, \tau_{\varepsilon_i}\}$ on $\widetilde{\sigma}^2$. The main results are summarised below.

Proposition 4 *Suppose both voters and parties believe that each ε_i is independently drawn from $N\left(0, \tau_{\varepsilon_i}^{-1}\right)$ for all i. Suppose they disagree on the precision of the hidden state and the bias terms, i.e., $\tau_s \neq \widehat{\tau}_s$ and $\tau_{b_i} \neq \widehat{\tau}_{b_i}$ for all i.*

(a) *Holding other factors constant, an increase in either τ_{b_i} or τ_{ε_i} will raise the value of ψ.*

(b) *Holding other factors constant, an increase in $\widehat{\tau}_{b_i}$ will lower $var_p(\widehat{m})$ and $\widetilde{\sigma}^2$.*

(c) *Suppose $\tau_{b_i} = \tau_b$, $\widehat{\tau}_{b_i} = \widehat{\tau}_b$ and $\tau_{\varepsilon_i} = \tau_\varepsilon$ for all i. Then, holding other factors constant, any changes in τ_b will have no effect on $var_p(\widehat{m})$, whereas an increase in τ_ε will lower $var_p(\widehat{m})$.*

(d) *Suppose $\tau_{b_i} = \tau_b$, $\widehat{\tau}_{b_i} = \widehat{\tau}_b$ and $\tau_{\varepsilon_i} = \tau_\varepsilon$ for all i. Then*

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} \gtreqless 0 \qquad \textit{if and only if} \qquad 2var_p(\widehat{m}) \gtreqless var(\widehat{m}).$$

Part (a) implies that an increase in either τ_{b_i} or τ_{ε_i} will promote the voters' confidence in the signals and enhance polarisation by strengthening the learning effect. Part (b) implies that an increase in any $\widehat{\tau}_{b_i}$ will reduce polarisation by suppressing the uncertainty effect. The effects of changing τ_{b_i} or τ_{ε_i} on $var_p(\widehat{m})$ are more difficult to determine. This is due to the fact that, in general, $var_p(\widehat{m})$ not only depends on the politicians' subjective belief, but also depends on $\{\tau_{b_1}, ..., \tau_{b_n}\}$, which is part of the voters' subjective belief. This feature is not found in the previous two cases. In order to obtain sharper results, we focus on a narrower case in which all n signals share the same precision parameters $\{\tau_{b_i}, \widehat{\tau}_{b_i}, \tau_{\varepsilon_i}\}$ in the last two parts of Proposition 4. Under this assumption, $var_p(\widehat{m})$ is independent of the voters' prior beliefs (i.e., τ_s and τ_b) as in the previous special cases. As a result, any increase in τ_b will have a positive effect on $\widetilde{\sigma}^2$ through ψ alone, which is similar to an increase in τ_s.

The last part of Proposition 4 states that, when $\{\tau_{b_i}, \widehat{\tau}_{b_i}, \tau_{\varepsilon_i}\}$ are identical across signals, any changes in signal precision will have the same effect as in part (b) of Propositions 2 and 3. The overall effect on $\widetilde{\sigma}^2$ is again determined by the relative magnitude between $2var_p(\widehat{m})$ and $var(\widehat{m})$. In the current case, $2var_p(\widehat{m}) \gtrless var(\widehat{m})$ if and only if

$$\tau_s \gtrless \frac{n\widehat{\tau}_s}{2n + \widehat{\tau}_s\left(2\widehat{\tau}_b^{-1} + \tau_\varepsilon^{-1} - \tau_b^{-1}\right)}.$$

In the absence of any disagreement between voters and politicians, i.e., when $\widehat{\tau}_s = \tau_s$ and $\widehat{\tau}_b = \tau_b$, improving signal precision will always lead to a greater extent of policy polarisation. If $\widehat{\tau}_b = \tau_b$ but $\widehat{\tau}_s \neq \tau_s$, then

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} \gtrless 0 \quad \text{if and only if} \quad \tau_s \gtrless \frac{n\widehat{\tau}_s}{2n + \widehat{\tau}_s\left(\tau_b^{-1} + \tau_\varepsilon^{-1}\right)}.$$

Likewise, if $\widehat{\tau}_s = \tau_s$ but $\widehat{\tau}_b \neq \tau_b$, then

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} \gtrless 0 \quad \text{if and only if} \quad \tau_b \gtrless \frac{\tau_s}{n + \tau_s\left(2\widehat{\tau}_b^{-1} + \tau_\varepsilon^{-1}\right)}.$$

In both cases, disagreement opens up the possibility that a more precise signal will make polarisation less likely and less severe. This happens when the voters are more ignorant about the underlying policy issue (i.e., τ_s is significantly lower than $\widehat{\tau}_s$) or when they have low confidence in the impartiality of the signal (i.e., a sufficiently low value of τ_b).

Case 4: Biased Signals (II)

We now revisit the case in which there is only one biased signal. Both voters and parties believe that the bias term is correlated with the hidden state. The covariance matrices of (s, b) in the voters' and parties' beliefs are, respectively, denoted by

$$\mathbf{\Sigma}_0 = \begin{bmatrix} \sigma_s^2 & \rho_{s,b}\sigma_s\sigma_b \\ \rho_{s,b}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix} \quad \text{and} \quad \widehat{\mathbf{\Sigma}}_0 = \begin{bmatrix} \widehat{\sigma}_s^2 & \widehat{\rho}_{s,b}\widehat{\sigma}_s\widehat{\sigma}_b \\ \widehat{\rho}_{s,b}\widehat{\sigma}_s\widehat{\sigma}_b & \widehat{\sigma}_b^2 \end{bmatrix}. \tag{28}$$

Suppose $\mu_s = \widehat{\mu}_s = 0$ and $\mu_b = \widehat{\mu}_b = 0$. Then the mean and variance of s in the voters' posterior beliefs are

$$E(s \mid m) = \underbrace{\left[\frac{Cov(s,m)}{var(m)}\right]}_{\psi} m,$$

$$var(s \mid m) = \sigma_s^2 - \frac{[Cov(s,m)]^2}{var(m)},$$

where $Cov(s,m) = \sigma_s^2 + \rho_{s,b}\sigma_s\sigma_b$ and $var(m) = \sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2 + 2\rho_{s,b}\sigma_s\sigma_b$. In all the previous cases, the responsiveness coefficient ψ is always strictly positive. In the current case, ψ can be either positive or negative depending on the sign of $Cov(s,m)$, which in turn depends on $\rho_{s,b}$. Specifically,

$$\psi \gtrless 0 \quad \text{if and only if} \quad \rho_{s,b} \gtrless -\frac{\sigma_s}{\sigma_b}.$$

A negative ψ means that voters will update their beliefs in the *opposite* direction as suggested by the signal. Note that this type of learning is possible only when $\rho_{s,b} \neq 0$ and $\sigma_b^2 > 0$. The sign of ψ, however, does not affect the parties' perceived uncertainty because

$$\widetilde{\sigma}^2 = \left[\frac{Cov(s,m)}{var(m)}\right]^2 var_p(m),$$

where $var_p(m) = \widehat{\sigma}_s^2 + \widehat{\sigma}_b^2 + \sigma_\varepsilon^2 + 2\widehat{\rho}_{s,b}\widehat{\sigma}_s\widehat{\sigma}_b$. In the current context, learning effect refers to an increase in polarisation brought by an increase in $|\psi|$ or ψ^2. Proposition 5 summarises the effects of the precision parameters $\{\tau_s, \tau_b, \widehat{\tau}_s, \widehat{\tau}_b, \tau_\varepsilon\}$ and the correlation parameters $\{\rho_{s,b}, \widehat{\rho}_{s,b}\}$ on $\widetilde{\sigma}^2$.

Proposition 5 *Suppose there is only one biased signal and the covariance matrices of (s,b) in the voters' and parties' beliefs are given by those in (28).*

(a) *Holding other factors constant,*

$$\frac{d\widetilde{\sigma}^2}{d\rho_{s,b}} \gtrless 0 \quad \textit{if and only if} \quad Cov(s,m)\left[var(m) - 2Cov(s,m)\right] \gtrless 0. \tag{29}$$

(b) *Holding other factors constant,*

$$\frac{d\widetilde{\sigma}^2}{d\tau_s} \gtrless 0 \qquad \textit{if and only if} \qquad \left(\rho_{s,b} + \frac{\sigma_s}{\sigma_b}\right)\left[\rho_{s,b} + \frac{2\left(\sigma_b^2 + \sigma_\varepsilon^2\right)}{\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2\right)}\frac{\sigma_s}{\sigma_b}\right] \lessgtr 0. \tag{30}$$

(c) *Holding other factors constant,*

$$\frac{d\widetilde{\sigma}^2}{d\tau_b} \gtrless 0 \qquad \textit{if and only if} \qquad \left(\rho_{s,b} + \frac{\sigma_s}{\sigma_b}\right)\left[\left[\rho_{s,b}\left(\sigma_s^2 + \sigma_b^2 - \sigma_\varepsilon^2\right) + 2\sigma_s\sigma_b\right]\right] \gtrless 0. \tag{31}$$

(d) *For any* $z \in \left\{\widehat{\tau}_s, \widehat{\tau}_b, \widehat{\rho}_{s,b}\right\}$,

$$\frac{d\widetilde{\sigma}^2}{dz} \gtrless 0 \qquad \textit{if and only if} \qquad \frac{dvar_p\left(s,m\right)}{dz} \gtrless 0. \tag{32}$$

(e) *Holding other factors constant,*

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} \gtrless 0 \qquad \textit{if and only if} \qquad 2var_p\left(m\right) \gtrless var\left(m\right).$$

The first three parts of Proposition 5 consider the effects of changing any $z \in \left\{\rho_{s,b}, \tau_s, \tau_b\right\}$ on $\widetilde{\sigma}^2$. Since these parameters are related to the voters' subjective prior belief, they will only affect ψ but not $var_p\left(m\right)$. We begin with a heuristic discussion on the conditions in (29)-(31), which share the same root:

$$\frac{d\widetilde{\sigma}^2}{dz} \gtrless 0 \qquad \text{if and only if} \qquad \psi\frac{d\psi}{dz} \gtrless 0.$$

This states that an increase in any $z \in \left\{\rho_{s,b}, \tau_s, \tau_b\right\}$ will promote polarisation if and only if it intensifies the learning effect (i.e., making ψ more positive or more negative). Contrarily, such an increase will lessen polarisation if and only if it suppresses the learning effect by moving ψ closer to zero. The effect of z on ψ can be further decomposed according to

$$\frac{d\psi}{dz} = \psi\left[\frac{1}{Cov\left(s,m\right)}\frac{dCov\left(s,m\right)}{dz} - \frac{1}{var\left(m\right)}\frac{dvar\left(m\right)}{dz}\right].$$

This captures the idea that any changes in z will affect both $Cov\left(s,m\right)$ and $var\left(m\right)$, thus leading to two (potentially opposing) effects on ψ. As an illustration, we will explain the various effects in detail for the case of $\rho_{s,b}$.

Recall that a positive (negative) value of $\rho_{s,b}$ means that the bias term tends to complement (contradict) the effect of s. Thus, a more positive (more negative) value is associated with both a larger (smaller) spread in the observed signal and a larger (smaller, or even negative) covariance between s and m. A higher *absolute* value of $Cov(s,m)$ means that voters are now more responsive to the signals, which intensify the learning effect and encourage polarisation. On the other hand, a higher value of $var(m)$ signifies a deterioration in signal quality, which weakens the learning effect. The net effect on polarisation depends on (i) the sign of $Cov(s,m)$, and (ii) whether $var(m)$ or $Cov(s,m)$ is more sensitive to $\rho_{s,b}$. These factors are encapsulated in (29), which covers three main scenarios: First, if $Cov(s,m) < 0$ then an increase in $\rho_{s,b}$ will lower $\widetilde{\sigma}^2$ and lessen polarisation. Recall that $Cov(s,m) < 0$ is equivalent to $\rho_{s,b} < -\sigma_s/\sigma_b$ and $\psi < 0$.[33] As $\rho_{s,b}$ increases toward $-\sigma_s/\sigma_b$, both $Cov(s,m)$ and ψ will approach zero which means there is nothing to learn about s from m. As a result, the median voter's policy ideal $E(s \mid m)$ will converge to a deterministic constant (zero). This will incentivise the two parties to choose the same policy position and eliminate polarisation. The second scenario is when $2Cov(s,m) > var(m) > 0$, which is equivalent to $\sigma_s^2 > \left(\sigma_b^2 + \sigma_\varepsilon^2\right)$ and $\psi > 1/2$. In this case, increasing $\rho_{s,b}$ will have a larger positive effect on $var(m)$ than on $Cov(s,m)$. This presses ψ toward the lower bound $1/2$ and reduces polarisation. Finally, if $var(m) > 2Cov(s,m) > 0$, or equivalently $1/2 > \psi > 0$, then an increase in $\rho_{s,b}$ will induce a stronger positive effect on $Cov(s,m)$ than on $var(m)$ and raise the value of ψ. Overall, these results describe a non-monotonic relationship between $\rho_{s,b}$ and polarisation. In particular, a more exaggerating bias term (i.e., a higher positive value of $\rho_{s,b}$) does not necessarily lead to more polarisation. It also depends on other confounding factors such as $\left\{\sigma_s^2, \sigma_b^2, \sigma_\varepsilon^2\right\}$.

The effects of τ_s and τ_b on $\widetilde{\sigma}^2$ can be interpreted along the same line. But their effects will also depend on the sign of $\rho_{s,b}$ and other parameter values which vastly expands the number of possible cases. In general, if $\rho_{s,b} > 0$, then an increase in τ_s will unambiguously reduce the learning effect and suppress polarisation. This is consistent with the findings in the previous cases. If $\rho_{s,b} > 0$ and $\sigma_s^2 + \sigma_b^2 > \sigma_\varepsilon^2$ are both satisfied, then an increase in τ_b will have the same effect.

Part (d) of Proposition 5 examines the effects of changing $\left\{\widehat{\tau}_s, \widehat{\tau}_b, \widehat{\rho}_{s,b}\right\}$ on $\widetilde{\sigma}^2$. Since these represent different aspects of the politicians' subjective prior belief, any changes in these parameters will affect $\widetilde{\sigma}^2$ through $var_p(m)$ alone and has no impact on ψ. Using $var_p(m) = \widehat{\sigma}_s^2 + \widehat{\sigma}_b^2 + \sigma_\varepsilon^2 + 2\widehat{\rho}_{s,b}\widehat{\sigma}_s\widehat{\sigma}_b$, we can get

$$\frac{dvar_p(m)}{d\widehat{\tau}_s} = -\left(\widehat{\sigma}_s + \widehat{\rho}_{s,b}\widehat{\sigma}_b\right)\widehat{\sigma}_s^3,$$

$$\frac{dvar_p(m)}{d\widehat{\tau}_b} = -\left(\widehat{\sigma}_b + \widehat{\rho}_{s,b}\widehat{\sigma}_s\right)\widehat{\sigma}_b^3 \quad \text{and} \quad \frac{dvar_p(m)}{d\widehat{\rho}_{s,b}} = 2\widehat{\sigma}_s\widehat{\sigma}_b > 0.$$

[33] Obviously, this scenario can be ruled out by assuming $\sigma_s \geq \sigma_b$. There is, however, no *a priori* reason for (or against) this assumption. Hence, we consider this as one possible scenario.

These results, together with (32), imply the following: First, an increase in $\widehat{\rho}_{s,b}$ will unambiguously raise the value of $\widetilde{\sigma}^2$ through $var_p(m)$. Intuitively, this means the uncertainty effect will become stronger if the parties perceive the bias term as more exaggerating. Second, an increase in either $\widehat{\tau}_s$ or $\widehat{\tau}_b$ will lower $var_p(m)$ and $\widetilde{\sigma}^2$, provided that $\widehat{\rho}_{s,b}$ is not too negative, i.e.,

$$\widehat{\rho}_{s,b} > -\frac{\widehat{\sigma}_s}{\widehat{\sigma}_b}.$$

This condition is equivalent to $Cov_p(s,m) > 0$, i.e., the political parties believe that the signal is positively correlated with the hidden state. This shows that the findings in part (a) of Propositions 2 and 3, and part (b) of Proposition 4 can be extended to the more general case in which $\widehat{\rho}_{s,b} \neq 0$ and $Cov_p(s,m) \geq 0$. Finally, part (e) of Proposition 5 shows that our previous results regarding the effect of τ_ε on $\widetilde{\sigma}^2$ will continue to hold in this case, regardless of the sign of $\rho_{s,b}$.

4 Welfare Analysis

In this section we focus on the *ex ante* welfare (i.e., welfare before the realisation of the signals) of an arbitrary voter in a polarised equilibrium. We present two sets of results. The first one concerns the welfare implications of parties' ideological differences in a polarised equilibrium. The second set of results concerns the welfare effects of signal quality improvement. In both instances, the disagreement between voters' and politicians' beliefs plays a crucial role in shaping the results.

4.1 Ideological Polarisation

We start by deriving a measure of *ex ante* welfare, which is a single arbitrary voter's expected utility based on her prior belief. Similar to the equilibrium analysis, we maintain the assumption that all voters share the same prior belief as the median voter. In all the cases considered in Section 3, the median voter's ideal policy position is determined by $E(s \mid \mathbf{m}) = \psi\widehat{m}$, where $\widehat{m}$ is a weighted average of the signals $\mathbf{m} = (m_1, m_2, ..., m_n)^T$. Since the signals are jointly normally distributed with a zero mean vector, the sufficient statistic $\widehat{m}$ is a normal random variable with mean zero. This is true even if the signals are correlated. The variance of $\widehat{m}$ under the voter's prior belief is denoted by $\sigma_m^2 \equiv var(\widehat{m})$. Let $G(\cdot)$ be the cumulative distribution function of $N(0, \sigma_m^2)$.

Upon observing the signals, voter v updates her belief according to (2) and (3). Conditional on $\mathbf{m}$, her expected utility if R wins is given by

$$\begin{aligned}
& E\left[U\left(x_{eq}^{*};\delta_v\right) \mid \mathbf{m}\right] \\
= & -E\left[\left(\delta_v + \psi\widehat{m} - x_{eq}^{*} + s - \psi\widehat{m}\right)^2 \mid \mathbf{m}\right] \\
= & \underbrace{-E\left[\left(\delta_v + \psi\widehat{m} - x_{eq}^{*}\right)^2 \mid \mathbf{m}\right]}_{\text{Expected utility under prediction}} - \underbrace{var\left(s \mid \mathbf{m}\right)}_{\text{Prediction error}}.
\end{aligned}$$

The first term is the expected utility based on the voter's prediction of s, i.e., $E\left(s \mid \mathbf{m}\right) = \psi\widehat{m}$. The second term is the prediction error, which is a constant according to (3). If L wins, then the above expression becomes

$$-E\left[\left(\delta_v + \psi\widehat{m} + x_{eq}^{*}\right)^2 \mid \mathbf{m}\right] - var\left(s \mid \mathbf{m}\right).$$

In any symmetric equilibrium, R wins if $E\left(s \mid \mathbf{m}\right) = \psi\widehat{m} > \overline{x} = 0$ and L wins if $\psi\widehat{m} < 0$. Hence, before $\mathbf{m}$ is realised, the voter's expected utility is

$$\begin{aligned}
E\left[U\left(x_{eq}^{*};\delta_v\right)\right] &= -\int_0^{\infty} E\left[\left(\delta_v + \psi\widehat{m} - x_{eq}^{*}\right)^2 \mid \mathbf{m}\right] dG\left(\widehat{m}\right) \\
&\quad - \int_{-\infty}^{0} E\left[\left(\delta_v + \psi\widehat{m} + x_{eq}^{*}\right)^2 \mid \mathbf{m}\right] dG\left(\widehat{m}\right) - var\left(s \mid \mathbf{m}\right) \\
&= \left[2\sqrt{\frac{2}{\pi}}\psi\sigma_m - x_{eq}^{*}\right] x_{eq}^{*} - \left(\delta_v^2 + \tau_s^{-1}\right). \qquad (33)
\end{aligned}$$

The derivation of (33) is shown in the Appendix. In the convergent equilibrium, i.e., $x_{eq}^{*} = 0$, the voter's expected utility can be simplified to become

$$E\left[U\left(0;\delta_v\right)\right] = -\left(\delta_v^2 + \tau_s^{-1}\right). \qquad (34)$$

Combining (33) and (34) gives

$$E\left[U\left(x_{eq}^{*};\delta_v\right)\right] - E\left[U\left(0;\delta_v\right)\right] = \left[2\sqrt{\frac{2}{\pi}}\psi\sigma_m - x_{eq}^{*}\right] x_{eq}^{*}, \qquad (35)$$

which indicates the welfare gain or loss due to policy polarisation in equilibrium. Based on this measure, polarisation is welfare-improving if and only if

$$0 \leq x_{eq}^{*} = \frac{2\phi - \gamma h\left(0\right)}{4h\left(0\right)\phi + 2} \leq 2\sqrt{\frac{2}{\pi}}\psi\sigma_m. \qquad (36)$$

The equality in the middle is the formula in (13). The welfare gain is at its highest level at the mid-point of this range, i.e., $x_{mid} = \sqrt{2/\pi}\psi\sigma_m$. A graphical illustration is shown in Figure 1.

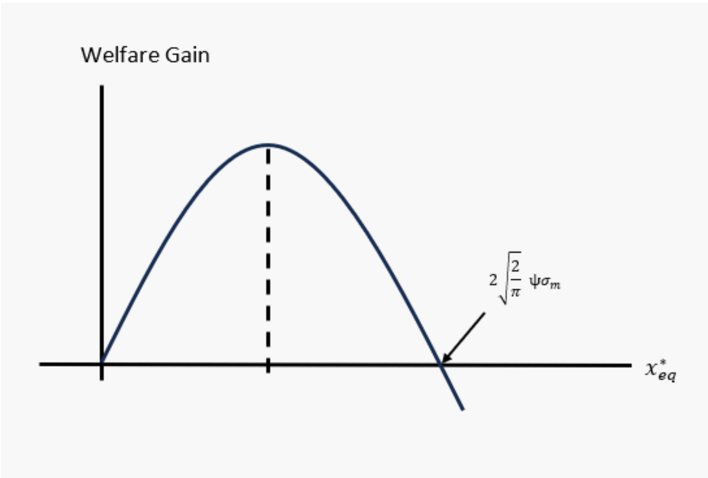

Figure 1: Welfare Gain from Polarisation.

The intuition behind Figure 1 is as follows. In the above discussion, the term $\psi\sigma_m = \sqrt{var\left(\psi\widehat{m}\right)}$ captures the voter's perceived uncertainty about the election outcome, which is determined by $E\left(s \mid \mathbf{m}\right) = \psi\widehat{m}$. The higher is this uncertainty, the greater the welfare gain from polarisation. This is because divergent policy platforms (i.e., $x_R \neq x_L$) can partially insure against the risk in election outcome faced by the voters, which explains why policy polarisation can improve the welfare of risk-averse voters. We refer to this as the *insurance effect* of policy polarisation. However, as the extent of policy divergence increases, further polarisation starts to reduce welfare. As shown in Figure 1, the net benefit of polarisation is positive and increasing when x^*_{eq} is below the mid-point $x_{mid} = \sqrt{2/\pi}\psi\sigma_m$, and decreasing when x^*_{eq} is above it. The diagram also shows that any further increase in polarisation will eventually turn the welfare gain into a welfare loss.

We are now ready to explore the welfare implications of parties' ideological differences. The first inequality in (36), which is derived in Proposition 1, states that polarised equilibrium exists only if the two parties' ideological differences are sufficiently large. In terms of our notations, these differences are captured by the parameter ϕ. It follows that

$$x^*_{eq} \geq 0 \qquad \text{if and only if} \qquad \phi \geq \frac{\gamma h\left(0\right)}{2} = \frac{\gamma}{\sqrt{2\pi}\widetilde{\sigma}} \equiv \phi_{\min}.$$

Our next result examines the conditions under which the second inequality in (36) is also satisfied.

Proposition 6

(i) *Suppose the following condition is valid,*

$$\frac{\widetilde{\sigma}}{\psi\sigma_m} \equiv \sqrt{\frac{var_p(\widehat{m})}{var(\widehat{m})}} \leq \frac{4}{\pi}.$$

Then polarisation is welfare-improving, i.e., $E\left[U\left(x_{eq}^*;\delta_v\right)\right] \geq E\left[U(0;\delta_v)\right]$, *for any* $x_{eq}^* \geq 0$, *or equivalently,* $\phi \geq \phi_{\min}$.

(ii) *Suppose the following condition is valid,*

$$\frac{\widetilde{\sigma}}{\psi\sigma_m} \equiv \sqrt{\frac{var_p(\widehat{m})}{var(\widehat{m})}} > \frac{4}{\pi}.$$

Then polarisation is welfare-improving if and only if

$$\phi_{\min} \leq \phi \leq \frac{\sqrt{\pi}\left(8\psi\sigma_m \cdot \widetilde{\sigma} + \gamma\right)}{2\sqrt{2}\left[\pi\widetilde{\sigma} - 4\psi\sigma_m\right]}. \tag{37}$$

Proposition 6 shows that whether polarisation is welfare-improving depends crucially on the interplay between two factors, namely (i) the disagreement between voters and politicians, and (ii) the ideological differences between the two parties. This can be explained as follows: From (13), it is evident that the extent of policy polarisation x_{eq}^* is strictly increasing in ϕ. As the ideological differences between the two parties continue to grow (i.e., as $\phi \to \infty$), x_{eq}^* will increase towards the limit

$$\lim_{\phi\to\infty} x_{eq}^* = \frac{1}{2h(0)} = \frac{\widetilde{\sigma}\sqrt{2\pi}}{2}.$$

This represents the maximum degree of polarisation possible under a given value of $\widetilde{\sigma}$, hence it is dependent on the parties' perceived variance $var_p(\widehat{m})$. If this limit falls within the range of positive welfare gain in Figure 1, i.e., $0 \leq \lim_{\phi\to\infty} x_{eq}^* \leq 2\sqrt{2/\pi}\psi\sigma_m$, then polarisation is always welfare-improving. Note that the upper boundary of this range is determined by the voters' perceived variance $var(\widehat{m})$. Thus, a comparison between $\lim_{\phi\to\infty} x_{eq}^*$ and $2\sqrt{2/\pi}\psi\sigma_m$ can be translated into a comparison between $var_p(\widehat{m})$ and $var(\widehat{m})$. Specifically,

$$\lim_{\phi\to\infty} x_{eq}^* = \frac{\widetilde{\sigma}\sqrt{2\pi}}{2} \lesseqgtr 2\sqrt{\frac{2}{\pi}}\psi\sigma_m \qquad \text{iff} \qquad \frac{\widetilde{\sigma}}{\psi\sigma_m} = \sqrt{\frac{var_p(\widehat{m})}{var(\widehat{m})}} \lesseqgtr \frac{4}{\pi}.$$

Holding ψ constant, as the voter becomes more uncertain about $\widehat{m}$ [i.e., when $var(\widehat{m})$ increases], the insurance effect of policy divergence will become more pronounced which makes polarisation more beneficial to the voters. In terms of Figure 1, this will take the form of an expansion in the range of positive welfare gain. On the other hand, when the parties become more uncertain about the election outcome [i.e., when $var_p(\widehat{m})$ increases], they will have a greater incentive to polarise which raise the value of x^*_{eq}. The first part of Proposition 6 states that if $var(\widehat{m})$ is sufficiently large relative to $var_p(\widehat{m})$, then all voters will be strictly better off in a society with highly partisan political parties ($\phi > \phi_m$) and policy polarisation ($x^*_{eq} > 0$) to an otherwise identical society but with more congruent parties ($\phi < \phi_{\min}$) and policy convergence ($x^*_{eq} = 0$). The second part of the proposition states that if $var_p(\widehat{m})$ is sufficiently larger than $var(\widehat{m})$, then the parties will have a stronger incentive to polarise but the benefits of policy divergence to the voters are modest. It follows that if both disagreement and the parties' ideological differences are large, i.e.,

$$\sqrt{\frac{var_p(\widehat{m})}{var(\widehat{m})}} > \frac{4}{\pi} \qquad \text{and} \qquad \phi > \frac{\sqrt{\pi}\,(8\psi\sigma_m \cdot \widetilde{\sigma} + \gamma)}{2\sqrt{2}\,[\pi\widetilde{\sigma} - 4\psi\sigma_m]},$$

then policy convergence (i.e., $x^*_{eq} = 0$) will be favoured by all voters and any equilibrium with $x^*_{eq} > 0$ is suboptimal.[34]

4.2 Improvement in Signal Quality

We now consider the welfare implications of an improvement in signal quality. Such an improvement can take any one of the following forms: (i) an increase in τ_{ε_i} in Case 1, for any $i \in \{1, 2, ..., n\}$; (ii) an increase in τ_ε or a decrease in ρ in Case 2; (iii) an increase in either τ_ε or τ_b in Case 3, under the assumption that all signals share the same $(\tau_\varepsilon, \tau_b)$; or (iv) an increase in τ_ε in Case 4. In the absence of disagreement, all such changes will raise the value of $\widetilde{\sigma}$. We will collectively represent this as

$$\frac{d\widetilde{\sigma}}{dz} > 0, \tag{38}$$

where z corresponds to τ_{ε_i}, τ_ε, τ_b or $-\rho$ depending on the specific case considered.

From (34), it is clear that any changes in z will have no impact on welfare in the convergent equilibrium. Our next result shows that, when voters' and politicians' beliefs align, then better signal quality will improve all voters' welfare in any polarised equilibrium.

[34] Our Proposition 6 is similar in spirit to Proposition 8 in Bernhardt *et al.* (2009, p.578). However, in their model the parties' perceived uncertainty about the median voter's policy preference (i.e., $\widetilde{\sigma}$) is an exogenous parameter and there is no disagreement between voters' and parties' beliefs.

Proposition 7 *Suppose there is no disagreement between voters' and politicians' beliefs, i.e.,* $\mathbf{\Sigma}_0 = \widehat{\mathbf{\Sigma}}_0$ *and* $var(\widehat{m}) = var_p(\widehat{m})$. *Then any improvement in signal quality (as described above) will unanimously improve voters' welfare in any polarised equilibrium, i.e.,*

$$\frac{dE\left[U\left(x_{eq}^{*};\delta_v\right)\right]}{dz} > 0, \qquad \text{for any } x_{eq}^{*} > 0 \text{ and for all } \delta_v.$$

A graphical illustration of this result is shown in Figure 2. To fix ideas, consider an increase in τ_{ε_i} in Case 1, for some $i \in \{1, 2, ..., n\}$. In the absence of disagreement, $\widetilde{\sigma}^2$ is the same as $\psi^2\sigma_m^2$. As we have seen in Proposition 2, more precise signals will always increase the value of $\widetilde{\sigma}^2$ in this case due to a dominating learning effect. This has two implications: First, an increase in $\psi\sigma_m$ will strengthen the insurance effect of polarisation and expand the range over which polarisation is welfare-improving. Second, an increase in $\widetilde{\sigma}^2$ will incentivise the parties to polarise and lead to an increase in x_{eq}^*. In the absence of disagreement, it can be shown that x_{eq}^* is always lower than the mid-point $x_{mid} = \sqrt{2/\pi}\psi\sigma_m$ (i.e., on the upward-sloping side of the curves both before and after the increase in τ_{ε_i}). This means that increasing polarisation is always welfare-improving when there is no disagreement.

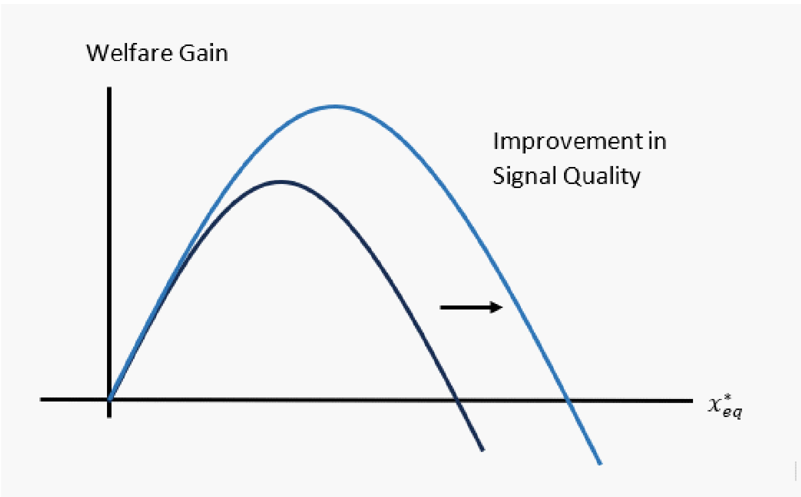

Figure 2: Improvement in Signal Quality.

The above result, however, may not hold when there is significant disagreement between voters and politicians. We demonstrate this possibility through two sets of numerical examples with a single unbiased signal. Set $\phi = 1$ and $\gamma = 3$ so that $\sigma_{\min} = 0.60$. In the first set of examples, we consider three combinations of τ_s and $\widehat{\tau}_s$, namely $(\tau_s, \widehat{\tau}_s) = (0.06, 0.60)$, $(\tau_s, \widehat{\tau}_s) = (0.06, 0.20)$ and $(\tau_s, \widehat{\tau}_s) = (0.06, 0.06)$. In the first two scenarios, disagreement exists and $\widehat{\tau}_s$ is much greater than

τ_s. According to part (c) of Proposition 2, more precise signal may lower the parties' perceived uncertainty and reduce polarisation when $\widehat{\tau}_s \gg \tau_s$. In the third scenario, voters' and politicians' beliefs coincide. Our theoretical results predict that in this case, any improvement in signal precision will unambiguously increase perceived uncertainty. These predictions are verified in Figure 3. The three diagrams on the left plot the value of $\widetilde{\sigma}$ over a range of τ_ε in these three cases. We see that in the first two cases, higher signal precision will lead to a reduction in perceived uncertainty and also policy polarisation.[35]

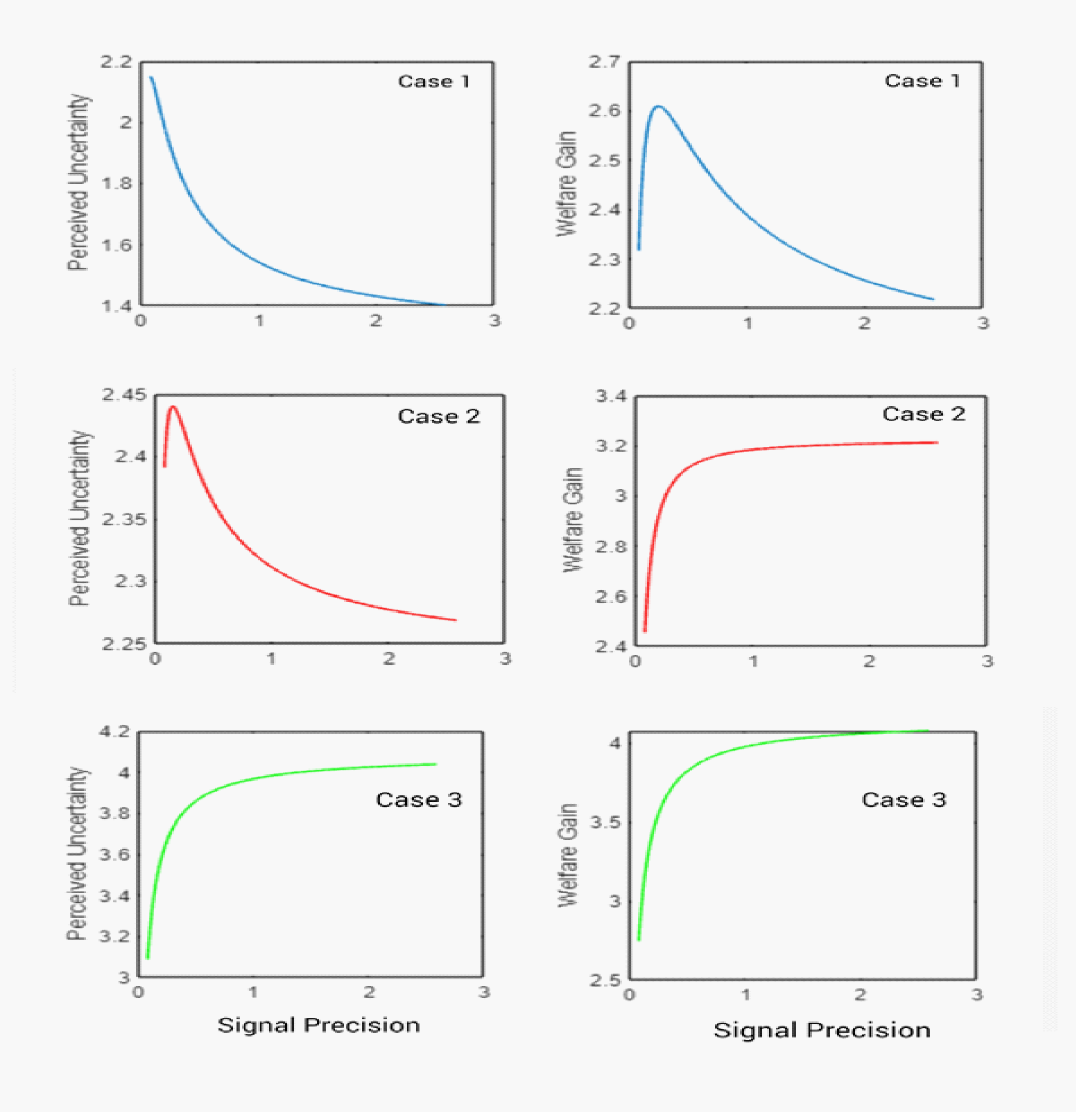

Figure 3: Results from Numerical Example 1.

The three diagrams on the right show the corresponding changes in the welfare gain mea-

[35] In all the cases that we considered, $\widetilde{\sigma}$ is greater than the threshold $\sigma_{\min} = 0.60$ so that x^*_{eq} is always strictly positive. As shown in Corollary 1, x^*_{eq} is strictly increasing in $\widetilde{\sigma}$ when $x^*_{eq} > 0$. Hence, a plot of x^*_{eq} against τ_ε will have the same shape as those depicted in the left column of Figure 3.

sure, $\left\{E\left[U\left(x_{eq}^*;\delta_v\right)\right]-E\left[U\left(0;\delta_v\right)\right]\right\}$. The uppermost panel shows that when $\widehat{\tau}_s$ is much greater than τ_s, signal quality improvement can be welfare-reducing. This can be explained as follows: Differentiating the expression in (35) with respect to τ_ε gives

$$\begin{aligned}&\frac{d}{d\tau_\varepsilon}\left\{E\left[U\left(x_{eq}^*;\delta_v\right)\right]-E\left[U\left(0;\delta_v\right)\right]\right\}\\ =\;&\underbrace{2\left(\sqrt{\frac{2}{\pi}}\psi\sigma_m-x_{eq}^*\right)}_{A}\frac{dx_{eq}^*}{d\tau_\varepsilon}+\underbrace{2\sqrt{\frac{2}{\pi}}\frac{d\left(\psi\sigma_m\right)}{d\tau_\varepsilon}x_{eq}^*}_{(+)}.\end{aligned} \tag{39}$$

The first term in the second line captures the effect of changing polarisation on voter welfare. As shown in Figure 1, the welfare gain from polarisation increases as x_{eq}^* approaches $x_{mid}=\sqrt{2/\pi}\psi\sigma_m$ from either side. Hence, the first effect is positive if an improvement in signal precision brings x_{eq}^* closer to $\sqrt{2/\pi}\psi\sigma_m$ [e.g., if $x_{eq}^*<\sqrt{2/\pi}\psi\sigma_m$ and $dx_{eq}^*/d\tau_\varepsilon>0$]. The second term represents the effect due to a change in voters' perceived uncertainty on election outcome ($\psi\sigma_m$). This essentially captures the same kind of movement as depicted in Figure 2. Hence this term is always positive whenever $x_{eq}^*>0$.

In all three cases, equilibrium policy x_{eq}^* is far lower than $x_{mid}=\sqrt{2/\pi}\psi\sigma_m$ over the range of τ_ε that we consider. Hence, the term that we label as A in (39) is always positive. In Case 3, x_{eq}^* is strictly increasing in τ_ε. This means any increase in τ_ε will bring x_{eq}^* closer to $\sqrt{2/\pi}\psi\sigma_m$ and increase the welfare gain from polarisation, hence

$$\left(\sqrt{\frac{2}{\pi}}\psi\sigma_m-x_{eq}^*\right)\frac{dx_{eq}^*}{d\tau_\varepsilon}>0.$$

This, together with the positive second term, leads to an unambiguous increase in welfare gain. This confirms the result in Proposition 7. On the other hand, x_{eq}^* is strictly decreasing in τ_ε in Case 1. This means any increase in τ_ε will bring x_{eq}^* further away $\sqrt{2/\pi}\psi\sigma_m$ and reduce the welfare gain, i.e.,

$$\left(\sqrt{\frac{2}{\pi}}\psi\sigma_m-x_{eq}^*\right)\frac{dx_{eq}^*}{d\tau_\varepsilon}<0. \tag{40}$$

The tug-of-war between this and the positive second term then contributes to the hump shape in the top-right diagram. As suggested by the diagram, the negative effect eventually dominates when τ_ε is sufficiently large. The middle-right diagram of Figure 3 can be explained along the same line.[36]

[36] These results are robust to a wide range of values of $(\phi,\gamma,\tau_s,\widehat{\tau}_s)$, hence it is easy to construct other examples that can deliver the same messages. We do not present the robustness checks here due to space consideration. The

Our second numerical example shows that signal quality improvement can also be welfare-reducing when $\tau_s \gg \widehat{\tau}_s$. Specifically, we set $\phi = 5$, $\gamma = 3$ and $(\tau_s, \widehat{\tau}_s) = (0.6, 0.06)$. Figure 4 shows how $\widetilde{\sigma}^2$ and $E\left[U\left(x^*_{eq}; \delta_v\right)\right] - E\left[U\left(0; \delta_v\right)\right]$ change over a range of τ_ε. In this case, x^*_{eq} is greater than $x_{mid} = \sqrt{2/\pi}\psi\sigma_m$ (i.e., on the downward sloping side of the parabola in Figure 1) and is strictly increasing in τ_ε [as suggested by Proposition 2 part (c)]. Thus, better signal precision will bring x^*_{eq} further away from $\sqrt{2/\pi}\psi\sigma_m$ and lower the welfare gain, i.e., (40) will hold. The diagram on the right suggests that this negative term dominates the positive second term in (39) so that welfare is strictly decreasing in τ_ε.

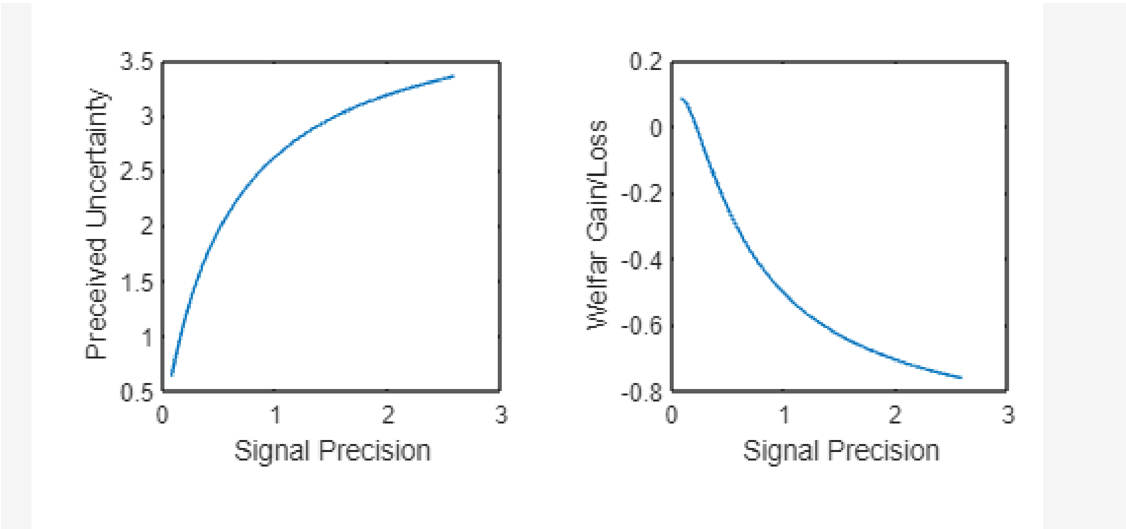

Figure 4: Results from Numerical Example 2.

5 Conclusion

The main objective of this paper is to examine how voters' political information processing and belief formation will affect political parties' strategic policy choices. To this end, we extend the canonical electoral competition model of Bernhardt *et al.* (2009) by introducing two new features, namely (i) perceived biasedness of the information sources, and (ii) disagreement between voters' and politicians' beliefs. Both are empirically relevant and we show that they can generate new results and insights. For instance, adding a random bias opens up the possibility of what we called "defiant learning." On the other hand, allowing for disagreement between voters' and politicians' beliefs can overturn the conventional wisdom that better signal precision will always promote polarisation and is always welfare-improving. Both empirical evidence and causal observations

MATLAB codes for generating the numerical results are available from the authors' personal website.

suggest that disagreement is ubiquitous in the political arena. In this paper, we only explore one form of disagreement (between politicians and their constituents). A broader investigation on how other forms of disagreement (e.g., divergence in opinions and beliefs among voters) will affect voters' belief formation and polarisation may be a fruitful avenue for future research.

Appendix

Proof of Lemma 1

The proof is based on a well-known result concerning conditional multivariate normal distributions which is stated as follows [see, for instance, Greene (2012, p.1042, Theorem B.7)]. Suppose $[\mathbf{X}_1, \mathbf{X}_2]$ has a joint multivariable normal distribution $\mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}.$$

The marginal distribution of $\mathbf{X}_i$ is given by $\mathbf{N}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_{ii})$ for $i \in \{1, 2\}$. Then the conditional distribution of $\mathbf{X}_1$ given $\mathbf{X}_2$ is normal with mean vector

$$\boldsymbol{\mu}_{1,2} = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2),$$

and covariance matrix

$$\boldsymbol{\Sigma}_{11,2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}.$$

In order to apply this result, first note that $(s, \mathbf{b}, \mathbf{m})$ has a joint multivariate normal distribution with mean vector $\boldsymbol{\mu}^\dagger$ and covariance matrix $\boldsymbol{\Sigma}^\dagger$ given by

$$\boldsymbol{\mu}^\dagger = \begin{bmatrix} \mu_s \\ \boldsymbol{\mu}_b \\ \boldsymbol{\mu}_m \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\Sigma}^\dagger = \begin{bmatrix} \sigma_s^2 & \boldsymbol{\Omega} & \boldsymbol{\Lambda} \\ \boldsymbol{\Omega}^T & \boldsymbol{\Sigma}_b & \boldsymbol{\Theta} \\ \boldsymbol{\Lambda}^T & \boldsymbol{\Theta}^T & \boldsymbol{\Sigma}_m \end{bmatrix}.$$

The meaning of $\boldsymbol{\Lambda}$ in the covariance matrix has been explained in the main text. The covariances between $\mathbf{b}$ and $\mathbf{m}$ are captured by the n-by-n matrix $\boldsymbol{\Theta} \equiv E\left[(\mathbf{b} - \boldsymbol{\mu}_\mathbf{b})(\mathbf{m} - \boldsymbol{\mu}_m)^T\right]$. The (i, j)th element of $\boldsymbol{\Theta}$ is denoted by $\theta_{i,j} \equiv Cov(b_i, m_j) = \omega_i + Cov(b_i, b_j)$.

Using the theorem mentioned above, the posterior distribution of $(s, \mathbf{b})$ after observing $\mathbf{m}$ is a normal distribution with mean vector

$$\boldsymbol{\mu}' = \begin{bmatrix} \mu_s \\ \boldsymbol{\mu}_b \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Lambda} \\ \boldsymbol{\Theta} \end{bmatrix} \boldsymbol{\Sigma}_m^{-1}(\mathbf{m} - \boldsymbol{\mu}_m), \tag{41}$$

and covariance matrix

$$\mathbf{\Sigma}' = \begin{bmatrix} \sigma_s^2 & \mathbf{\Omega} \\ \mathbf{\Omega}^T & \mathbf{\Sigma}_b \end{bmatrix} - \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{\Theta} \end{bmatrix} \mathbf{\Sigma}_m^{-1} \begin{bmatrix} \mathbf{\Lambda}^T & \mathbf{\Theta}^T \end{bmatrix}. \tag{42}$$

It follows that the marginal distribution of s in the voters' posterior belief is also normal. To derive the posterior mean and posterior variance of s, we first define $\kappa_{i,j}$ as the element on the ith row and jth column of $\mathbf{\Sigma}_m^{-1}$. Then

$$\begin{aligned}
\begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{\Theta} \end{bmatrix} \mathbf{\Sigma}_m^{-1} (\mathbf{m} - \boldsymbol{\mu}_m) &= \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{bmatrix} \begin{bmatrix} \kappa_{1,1} & \cdots & \kappa_{1,n} \\ \vdots & \ddots & \\ \kappa_{n,1} & \cdots & \kappa_{n,n} \end{bmatrix} \begin{bmatrix} m_1 - \mu_{m_1} \\ \vdots \\ m_n - \mu_{m_n} \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{bmatrix}}_{(n+1)\text{-by-}n} \underbrace{\begin{bmatrix} \sum_{j=1}^n \kappa_{1,j} \left(m_j - \mu_{m_j} \right) \\ \vdots \\ \sum_{j=1}^n \kappa_{n,j} \left(m_j - \mu_{m_j} \right) \end{bmatrix}}_{n\text{-by-}1}.
\end{aligned}$$

The first entry in the resulting $(n+1)$-by-1 vector is

$$\mathbf{\Lambda} \mathbf{\Sigma}_m^{-1} (\mathbf{m} - \boldsymbol{\mu}_m) = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \kappa_{i,j} \left(m_j - \mu_{m_j} \right).$$

It follows from (41) that the posterior mean of s is

$$\begin{aligned}
E(s \mid \mathbf{m}) &= \mu_s + \sum_{i=1}^n \sum_{j=1}^n \lambda_i \kappa_{i,j} \left(m_j - \mu_{m_j} \right) \\
&= \mu_s + \sum_{j=1}^n \underbrace{\left(\sum_{i=1}^n \lambda_i \kappa_{i,j} \right)}_{\alpha_j} \left(m_j - \mu_{m_j} \right).
\end{aligned}$$

Similarly,

$$\begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{\Theta} \end{bmatrix} \mathbf{\Sigma}_m^{-1} \begin{bmatrix} \mathbf{\Lambda}^T & \mathbf{\Theta}^T \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{bmatrix} \begin{bmatrix} \kappa_{1,1} & \cdots & \kappa_{1,n} \\ \vdots & \ddots & \\ \kappa_{n,1} & \cdots & \kappa_{n,n} \end{bmatrix} \begin{bmatrix} \lambda_1 & \theta_{1,1} & \cdots & \theta_{n,1} \\ \vdots & & & \vdots \\ \lambda_n & \theta_{1,n} & \cdots & \theta_{n,n} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^n \kappa_{1,j}\lambda_j & \sum_{j=1}^n \kappa_{1,j}\theta_{1,j} & \cdots & \sum_{j=1}^n \kappa_{1,j}\theta_{n,j} \\ \vdots & & & \vdots \\ \sum_{j=1}^n \kappa_{n,j}\lambda_j & \sum_{j=1}^n \kappa_{n,j}\theta_{1,j} & \cdots & \sum_{j=1}^n \kappa_{n,j}\theta_{n,j} \end{bmatrix}.$$

The $(1,1)$th element of the resulting $(n+1)$-by-$(n+1)$ matrix is

$$\mathbf{\Lambda}\mathbf{\Sigma}_m^{-1}\mathbf{\Lambda}^T = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \kappa_{i,j} \lambda_j.$$

It follows from (42) that the posterior variance of s is

$$var\,(s \mid \mathbf{m}) = \sigma_s^2 - \sum_{j=1}^n \left(\sum_{i=1}^n \lambda_i \kappa_{i,j} \right) \lambda_j.$$

This completes the proof of Lemma 1.

Proof of Lemma 2

Suppose each b_i, $i \in \{1, 2, ..., n\}$, is a deterministic constant normalised to zero, and suppose $\mu_s = 0$. Then $(s, \mathbf{m})$ has a joint multivariate normal distribution with zero mean vector and covariance matrix $\boldsymbol{V}$ given by

$$\boldsymbol{V} = \begin{bmatrix} \sigma_s^2 & \mathbf{\Lambda}^T \\ \mathbf{\Lambda} & \mathbf{\Sigma}_m \end{bmatrix},$$

where $\mathbf{\Lambda} = \sigma_s^2 \times \mathbf{1}_n$. Thus, for Case 1 and Case 2 where signals are unbiased, $\lambda_i = \sigma_s^2$ for all i.

Suppose each ε_i is drawn from the distribution $N\left(0, \sigma_{\varepsilon_i}^2\right)$, where $\sigma_{\varepsilon_i}^2 = \tau_{\varepsilon_i}^{-1}$. Then the covariance structure of $\{m_1, ..., m_n\}$ is given by

$$Cov\,(m_i, m_j) = \begin{cases} \sigma_s^2 + \sigma_{\varepsilon_i}^2 & \text{for } i = j, \\ \sigma_s^2 & \text{for } i \neq j. \end{cases}$$

Hence, $\boldsymbol{\Sigma}_m$ can be expressed as the sum of two n-by-n matrices,

$$\boldsymbol{\Sigma}_m = \mathbf{A} + \sigma_s^2 \mathbf{1}_n \mathbf{1}_n^T,$$

where $\mathbf{A}$ is a diagonal matrix with diagonal elements $\left(\sigma_{\varepsilon_1}^2, ..., \sigma_{\varepsilon_n}^2\right)$. The inverse of $\boldsymbol{\Sigma}_m$ can be derived using equation (3) in Henderson and Searle (1981, p.53). Specifically, this equation states that for any matrix $\mathbf{M} = \mathbf{A} + r\mathbf{u}\mathbf{v}^T$, where $\mathbf{A}$ can be any invertible matrix, r is a scalar, $\mathbf{u}$ is a column vector and $\mathbf{v}^T$ is a row vector, the inverse can be expressed as

$$\mathbf{M}^{-1} = \mathbf{A}^{-1} - \xi \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}, \tag{43}$$

where

$$\xi = \frac{r}{1 + r\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}.$$

Hence, by setting $r = \sigma_s^2$, $\mathbf{u} = \mathbf{1}_n$ and $\mathbf{v}^T = \mathbf{1}_n^T$, we can get

$$\boldsymbol{\Sigma}_m^{-1} = \mathbf{A}^{-1} - \xi \mathbf{A}^{-1}\mathbf{1}_n\mathbf{1}_n^T\mathbf{A}^{-1}, \tag{44}$$

where

$$\xi = \frac{\sigma_s^2}{1 + \sigma_s^2 \mathbf{1}_n^T\mathbf{A}^{-1}\mathbf{1}_n}.$$

Since $\mathbf{A}$ is a diagonal matrix, its inverse is simply

$$\mathbf{A}^{-1} = \begin{bmatrix} \tau_{\varepsilon_1} & 0 & \cdots & 0 \\ 0 & \tau_{\varepsilon_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \tau_{\varepsilon_n} \end{bmatrix}. \tag{45}$$

It follows that $\mathbf{1}_n^T\mathbf{A}^{-1}\mathbf{1}_n = \sum_{i=1}^n \tau_{\varepsilon_i}$, and

$$\xi = \frac{\sigma_s^2}{1 + \sigma_s^2 \sum_{i=1}^n \tau_{\varepsilon_i}} = \frac{1}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}}, \tag{46}$$

where $\tau_s \equiv \sigma_s^{-2}$. In addition,

$$\mathbf{A}^{-1}\mathbf{1}_n\mathbf{1}_n^T\mathbf{A}^{-1} = \begin{bmatrix} \tau_{\varepsilon_1} \\ \tau_{\varepsilon_2} \\ \vdots \\ \tau_{\varepsilon_n} \end{bmatrix} \begin{bmatrix} \tau_{\varepsilon_1} & \tau_{\varepsilon_2} & \cdots & \tau_{\varepsilon_n} \end{bmatrix} = \begin{bmatrix} \tau_{\varepsilon_1}^2 & \tau_{\varepsilon_1}\tau_{\varepsilon_2} & \cdots & \tau_{\varepsilon_1}\tau_{\varepsilon_n} \\ \tau_{\varepsilon_1}\tau_{\varepsilon_2} & \tau_{\varepsilon_2}^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \tau_{\varepsilon_1}\tau_{\varepsilon_n} & \cdots & \cdots & \tau_{\varepsilon_n}^2 \end{bmatrix}. \quad (47)$$

Using (44)-(47), we can express the elements on any jth column of $\mathbf{\Sigma}_m^{-1}$ as

$$\kappa_{i,j} = \begin{cases} \tau_{\varepsilon_j} - \xi {\tau_{\varepsilon_j}}^2 & \text{for } i = j, \\ -\xi\tau_{\varepsilon_i}\tau_{\varepsilon_j} & \text{for } i \neq j, \end{cases}$$

$$\Rightarrow \alpha_j = \sum_{i=1}^{n} \lambda_i \kappa_{i,j} = \sigma_s^2 \tau_{\varepsilon_j} \left(1 - \xi \sum_{i=1}^{n} \tau_{\varepsilon_i}\right) = \frac{\tau_{\varepsilon_j}}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}}.$$

Substituting these and $\lambda_i = \sigma_s^2$ into (2) and (3) gives

$$E(s \mid \mathbf{m}) = \sum_{i=1}^{n} \alpha_i m_i = \frac{\sum_{i=1}^{n} \tau_{\varepsilon_i} m_i}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} = \underbrace{\frac{\sum_{i=1}^{n} \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}}}_{\psi} \cdot \underbrace{\sum_{i=1}^{n} \zeta_i m_i}_{\widehat{m}},$$

where $\zeta_i \equiv \tau_{\varepsilon_i} / \sum_{i=1}^{n} \tau_{\varepsilon_i}$ for all i, and

$$var(s \mid \mathbf{m}) = \frac{1}{\tau_s} - \sum_{i=1}^{n} \lambda_i \alpha_i = \frac{1}{\tau_s} - \frac{1}{\tau_s} \frac{\sum_{i=1}^{n} \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} = \frac{1}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}}.$$

Finally, under the parties' beliefs, the covariance structure of $\{m_1, ..., m_n\}$ is given by

$$Cov_p(m_i, m_j) = \begin{cases} \widehat{\sigma}_s^2 + \sigma_{\varepsilon_j}^2 & \text{for } i = j, \\ \widehat{\sigma}_s^2 & \text{for } i \neq j, \end{cases}$$

for any given j, and the perceived uncertainty is given by

$$\widetilde{\sigma}^2 = \left(\frac{\sum_{i=1}^{n} \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}}\right)^2 var_p(\widehat{m}),$$

where

$$var_p(\widehat{m}) = \sum_{j=1}^{n} \zeta_j \sum_{i=1}^{n} \zeta_i Cov_p(m_i, m_j) = \sum_{j=1}^{n} \zeta_j \left(\widehat{\sigma}_s^2 \sum_{i=1}^{n} \zeta_i + \zeta_j \sigma_{\varepsilon_j}^2\right).$$

Since $\sum_{j=1}^{n} \zeta_j = 1$ and $\zeta_j \sigma_{\varepsilon_j}^2 = \left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)^{-1}$ for all j, we can simplify the above expression to become

$$var_p(\widehat{m}) = \widehat{\sigma}_s^2 + \left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)^{-1} = \frac{\left(\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)}{\widehat{\tau}_s\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)}. \tag{48}$$

Using the same line of argument and replacing $\widehat{\sigma}_s^2$ with σ_s^2, we can show that

$$var(\widehat{m}) = \frac{\left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)}{\tau_s\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)}, \tag{49}$$

which is the unconditional variance of $\widehat{m}$ under the voters' subjective prior belief. This completes the proof of Lemma 2.

Proof of Proposition 2

Recall that perceived uncertainty $\widetilde{\sigma}^2$ can be expressed as

$$\widetilde{\sigma}^2 = \underbrace{\left(\frac{\sum_{i=1}^{n} \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}}\right)^2}_{\psi^2} \cdot \underbrace{\frac{\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}}{\widehat{\tau}_s \sum_{i=1}^{n} \tau_{\varepsilon_i}}}_{var_p(\widehat{m})}.$$

It is clear that any changes in τ_s will only affect ψ but not $var_p(\widehat{m})$. Likewise, any changes in $\widehat{\tau}_s$ will only affect $var_p(\widehat{m})$ but not ψ. Consider the logarithm of ψ,

$$\ln\psi = \ln\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right) - \ln\left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right).$$

Totally differentiating this with respect to $\{\psi, \tau_s, \tau_{\varepsilon_i}\}$ gives

$$\frac{d\psi}{\psi} = -\frac{\tau_s}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} \frac{d\tau_s}{\tau_s} + \frac{\tau_s \tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)\left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)} \frac{d\tau_{\varepsilon_i}}{\tau_{\varepsilon_i}}.$$

Suppose $d\tau_{\varepsilon_i} = 0$, then we have

$$\frac{d\psi}{d\tau_s} = -\frac{\psi}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} < 0 \quad \Rightarrow \quad \frac{d\widetilde{\sigma}^2}{d\tau_s} < 0. \tag{50}$$

On the other hand, if $d\tau_s = 0$, then

$$\frac{d\psi}{d\tau_{\varepsilon_i}} = \frac{\tau_s \psi}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)\left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)} > 0, \tag{51}$$

$$\Rightarrow \frac{\tau_{\varepsilon_i}}{\psi} \frac{d\psi}{d\tau_{\varepsilon_i}} = \frac{\tau_{\varepsilon_i}}{\sum_{i=1}^{n} \tau_{\varepsilon_i}} \frac{\tau_s}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} = \frac{\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)^2} \frac{1}{var(\widehat{m})}.$$

The second equality follows from (49). Similarly, totally differentiating $\ln\left[var_p\left(\widehat{m}\right)\right]$ with respect to $\{\psi, \widehat{\tau}_s, \tau_{\varepsilon_i}\}$ gives

$$\ln\left[var_p\left(\widehat{m}\right)\right] = \ln\left[\widehat{\tau}_s + \sum_{i=1}^{n}\tau_{\varepsilon_i}\right] - \ln\widehat{\tau}_s - \ln\left(\sum_{i=1}^{n}\tau_{\varepsilon_i}\right)$$

$$\frac{dvar_p\left(\widehat{m}\right)}{var_p\left(\widehat{m}\right)} = -\frac{\sum_{i=1}^{n}\tau_{\varepsilon_i}}{\widehat{\tau}_s + \sum_{i=1}^{n}\tau_{\varepsilon_i}}\frac{d\widehat{\tau}_s}{\widehat{\tau}_s} - \frac{\widehat{\tau}_s\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n}\tau_{\varepsilon_i}\right)\left(\widehat{\tau}_s + \sum_{i=1}^{n}\tau_{\varepsilon_i}\right)}\frac{d\tau_{\varepsilon_i}}{\tau_{\varepsilon_i}}.$$

When all other factors except $\widehat{\tau}_s$ are kept constant,

$$\frac{dvar_p\left(\widehat{m}\right)}{d\widehat{\tau}_s} = -\frac{\sum_{i=1}^{n}\tau_{\varepsilon_i}}{\widehat{\tau}_s + \sum_{i=1}^{n}\tau_{\varepsilon_i}}\frac{var_p\left(\widehat{m}\right)}{\widehat{\tau}_s} < 0 \quad \Rightarrow \quad \frac{d\widetilde{\sigma}^2}{d\widehat{\tau}_s} < 0. \tag{52}$$

If $d\tau_s = 0$, then

$$\frac{dvar_p\left(\widehat{m}\right)}{d\tau_{\varepsilon_i}} = -\frac{\widehat{\tau}_s var_p\left(\widehat{m}\right)}{\left(\sum_{i=1}^{n}\tau_{\varepsilon_i}\right)\left(\widehat{\tau}_s + \sum_{i=1}^{n}\tau_{\varepsilon_i}\right)} < 0, \tag{53}$$

$$\Rightarrow \frac{\tau_{\varepsilon_i}}{var_p\left(\widehat{m}\right)}\frac{dvar_p\left(\widehat{m}\right)}{d\tau_{\varepsilon_i}} = -\frac{\tau_{\varepsilon_i}}{\sum_{i=1}^{n}\tau_{\varepsilon_i}}\frac{\widehat{\tau}_s}{\left(\widehat{\tau}_s + \sum_{i=1}^{n}\tau_{\varepsilon_i}\right)} = -\frac{\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n}\tau_{\varepsilon_i}\right)^2}\frac{1}{var_p\left(\widehat{m}\right)}.$$

The second equality follows from (48). Equations (51) and (53) together prove the statement in part (b).

Holding τ_s and $\widehat{\tau}_s$ constant, the overall effect of changing τ_{ε_i} on $\widetilde{\sigma}^2$ can be determined by

$$\begin{aligned}\frac{\tau_{\varepsilon_i}}{\widetilde{\sigma}^2}\frac{d\widetilde{\sigma}^2}{d\tau_{\varepsilon_i}} &= 2\frac{\tau_{\varepsilon_i}}{\psi}\frac{d\psi}{d\tau_{\varepsilon_i}} + \frac{\tau_{\varepsilon_i}}{var_p\left(\widehat{m}\right)}\frac{dvar_p\left(\widehat{m}\right)}{d\tau_{\varepsilon_i}} \\ &= \left[\frac{2}{var\left(\widehat{m}\right)} - \frac{1}{var_p\left(\widehat{m}\right)}\right]\frac{\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n}\tau_{\varepsilon_i}\right)^2}.\end{aligned}$$

Hence,

$$\frac{d\widetilde{\sigma}^2}{d\tau_{\varepsilon_i}} \gtrless 0 \Leftrightarrow 2var_p\left(\widehat{m}\right) \gtrless var\left(\widehat{m}\right).$$

Using (48) and (49), we can show that

$$2var_p\left(\widehat{m}\right) \gtrless var\left(\widehat{m}\right) \quad \text{if and only if} \quad \frac{2\tau_s}{\tau_s + \sum_{i=1}^{n}\tau_{\varepsilon_i}} \gtrless \frac{\widehat{\tau}_s}{\widehat{\tau}_s + \sum_{i=1}^{n}\tau_{\varepsilon_i}},$$

which is equivalent to

$$\tau_s \gtrless \frac{\widehat{\tau}_s\sum_{i=1}^{n}\tau_{\varepsilon_i}}{\widehat{\tau}_s + 2\sum_{i=1}^{n}\tau_{\varepsilon_i}}.$$

This completes the proof of Proposition 2.

Proof of Lemma 3

Suppose $\rho \geq -1/(n-1)$. The inverse of $\mathbf{\Sigma}_\varepsilon$ can be shown to take the following form

$$\mathbf{\Sigma}_\varepsilon^{-1} = \frac{\tau_\varepsilon}{1+(n-2)\rho-(n-1)\rho^2}\begin{bmatrix} 1+(n-2)\rho & -\rho & \cdots & -\rho \\ -\rho & 1+(n-2)\rho & \cdots & -\rho \\ \vdots & & \ddots & \vdots \\ -\rho & \cdots & -\rho & 1+(n-2)\rho \end{bmatrix}. \quad (54)$$

To see this, note that all diagonal entries of $\mathbf{\Sigma}_\varepsilon \mathbf{\Sigma}_\varepsilon^{-1}$ are given by

$$\frac{1}{1+(n-2)\rho-(n-1)\rho^2}\left[1+(n-2)\rho-(n-1)\rho^2\right] = 1,$$

and all off-diagonal elements of $\mathbf{\Sigma}_\varepsilon \mathbf{\Sigma}_\varepsilon^{-1}$ are given by

$$\frac{1}{1+(n-2)\rho-(n-1)\rho^2}\left\{-\rho+[1+(n-2)\rho]\rho+(n-2)\rho^2\right\} = 0.$$

Define the notation ν according to

$$\nu \equiv \frac{\tau_\varepsilon}{1+(n-2)\rho-(n-1)\rho^2} = \frac{\tau_\varepsilon}{(1-\rho)[1+(n-1)\rho]}.$$

The covariances among the signals $\{m_1, ..., m_n\}$ are given by $Cov(m_i, m_j) = \sigma_s^2 + Cov(\varepsilon_i, \varepsilon_j)$, which implies

$$\mathbf{\Sigma}_m = \mathbf{\Sigma}_\varepsilon + +\sigma_s^2 \mathbf{1}_n \mathbf{1}_n^T.$$

Using the same formula in (43), we can get

$$\mathbf{\Sigma}_m^{-1} = \mathbf{\Sigma}_\varepsilon^{-1} - \xi \mathbf{\Sigma}_\varepsilon^{-1} \mathbf{1}_n \mathbf{1}_n^T \mathbf{\Sigma}_\varepsilon^{-1}, \quad (55)$$

where

$$\xi = \frac{\sigma_s^2}{1+\sigma_s^2 \mathbf{1}_n^T \mathbf{\Sigma}_\varepsilon^{-1} \mathbf{1}_n} = \frac{1}{\tau_s + \mathbf{1}_n^T \mathbf{\Sigma}_\varepsilon^{-1} \mathbf{1}_n}.$$

It is straightforward to show that

$$\mathbf{1}_n^T \mathbf{\Sigma}_\varepsilon^{-1} \mathbf{1}_n = n\nu(1-\rho) = \frac{n\tau_\varepsilon}{1+(n-1)\rho}.$$

$$\Rightarrow \xi = \frac{1+(n-1)\rho}{n\tau_{\varepsilon}+\tau_s\left[1+(n-1)\rho\right]} \tag{56}$$

On the other hand,

$$\mathbf{\Sigma}_{\varepsilon}^{-1}\mathbf{1}_n\mathbf{1}_n^T\mathbf{\Sigma}_{\varepsilon}^{-1} = \nu^2(1-\rho)^2\,\mathbf{1}_n\mathbf{1}_n^T. \tag{57}$$

Using (55)-(57), we can write the elements on any jth column of $\mathbf{\Sigma}_m^{-1}$ as

$$\kappa_{i,j} = \begin{cases} \nu\left[1+(n-2)\rho\right]-\xi\nu^2(1-\rho)^2 & \text{for } i=j, \\ -\nu\rho-\xi\nu^2(1-\rho)^2 & \text{for } i\neq j. \end{cases}$$

Using these and $\lambda_i = \sigma_s^2 = \tau_s^{-1}$, we can get

$$\begin{aligned} \alpha_j &= \sum_{i=1}^{n}\lambda_i\kappa_{i,j} = \frac{1}{\tau_s}\left[\nu(1-\rho)-n\xi\nu^2(1-\rho)^2\right] \\ &= \frac{\tau_{\varepsilon}}{\tau_s\left[1+(n-1)\rho\right]}\left[1-\frac{n\tau_{\varepsilon}\xi}{1+(n-1)\rho}\right] \\ &= \frac{\tau_{\varepsilon}}{n\tau_{\varepsilon}+\tau_s\left[1+(n-1)\rho\right]}. \end{aligned}$$

Hence, the posterior mean and posterior variance of s are given by

$$E(s\mid\mathbf{m}) = \underbrace{\frac{n\tau_{\varepsilon}}{n\tau_{\varepsilon}+\tau_s\left[1+(n-1)\rho\right]}}_{\psi}\cdot\underbrace{\frac{1}{n}\sum_{i=1}^{n}m_i}_{\widehat{m}},$$

$$var(s\mid\mathbf{m}) = \frac{1}{\tau_s}\left(1-\sum_{i=1}^{n}\alpha_j\right) = \frac{1+(n-1)\rho}{n\tau_{\varepsilon}+\tau_s\left[1+(n-1)\rho\right]}.$$

From the parties' perspective, the covariance structure of $\{m_1,...,m_n\}$ is now given by

$$Cov_p(m_i,m_j) = \begin{cases} \widehat{\tau}_s^{-1}+\tau_{\varepsilon}^{-1} & \text{for } i=j, \\ \widehat{\tau}_s^{-1}+\tau_{\varepsilon}^{-1}\rho & \text{for } i\neq j, \end{cases} \tag{58}$$

and the perceived uncertainty is given by

$$\widetilde{\sigma}^2 = \left\{\frac{n\tau_{\varepsilon}}{n\tau_{\varepsilon}+\tau_s\left[1+(n-1)\rho\right]}\right\}^2 var_p(\widehat{m}),$$

where

$$\begin{aligned} var_p(\widehat{m}) &= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} Cov_p(m_i, m_j) \\ &= \frac{1}{n^2}\sum_{j=1}^{n}\left\{n\widehat{\tau}_s^{-1} + \tau_\varepsilon^{-1}[1+(n-1)\rho]\right\} \\ &= \frac{n\tau_\varepsilon + \widehat{\tau}_s[1+(n-1)\rho]}{n\tau_\varepsilon\widehat{\tau}_s}. \end{aligned} \tag{59}$$

Using the same steps, with $\widehat{\tau}_s^{-1}$ replaced by τ_s^{-1} in (58), we can show that

$$var(\widehat{m}) = \frac{n\tau_\varepsilon + \tau_s[1+(n-1)\rho]}{n\tau_\varepsilon\tau_s}. \tag{60}$$

This completes the proof of Lemma 3.

Proof of Proposition 3

Part (a) As shown above,

$$\widetilde{\sigma}^2 = \underbrace{\left\{\frac{n\tau_\varepsilon}{n\tau_\varepsilon + \tau_s[1+(n-1)\rho]}\right\}^2}_{\psi^2} \cdot \underbrace{\frac{n\tau_\varepsilon + \widehat{\tau}_s[1+(n-1)\rho]}{n\tau_\varepsilon\widehat{\tau}_s}}_{var_p(\widehat{m})}.$$

It is clear that any changes in τ_s will only affect ψ but not $var_p(\widehat{m})$. In particular, ψ (and hence $\widetilde{\sigma}^2$) is strictly decreasing in τ_s when $\rho > -1/(n-1)$. If $\rho = -1/(n-1)$, then ψ, $var_p(\widehat{m})$ and $\widetilde{\sigma}^2$ are all independent of τ_s. On the other hand, an increase in $\widehat{\tau}_s$ will lower $\widetilde{\sigma}^2$ because

$$var_p(\widehat{m}) = \frac{1}{\widehat{\tau}_s} + \frac{[1+(n-1)\rho]}{n\tau_\varepsilon},$$

which is strictly decreasing in $\widehat{\tau}_s$, and ψ is independent of $\widehat{\tau}_s$.

Part (b) Consider the logarithm of ψ and $var_p(\widehat{m})$,

$$\ln\psi = \ln n + \ln\tau_\varepsilon - \ln\{n\tau_\varepsilon + \tau_s[1+(n-1)\rho]\},$$

$$\ln[var_p(\widehat{m})] = \ln\{n\tau_\varepsilon + \widehat{\tau}_s[1+(n-1)\rho]\} - \ln n - \ln\tau_\varepsilon - \ln\widehat{\tau}_s.$$

Holding $\{\tau_s, \widehat{\tau}_s, \rho, n\}$ constant, consider the total derivatives of these with respect to τ_ε, i.e.,

$$\frac{d\psi}{\psi} = \frac{\tau_s\left[1+(n-1)\rho\right]}{n\tau_\varepsilon + \tau_s\left[1+(n-1)\rho\right]}\frac{d\tau_\varepsilon}{\tau_\varepsilon} = \frac{1+(n-1)\rho}{n\tau_\varepsilon var\left(\widehat{m}\right)}\frac{d\tau_\varepsilon}{\tau_\varepsilon}, \tag{61}$$

$$\frac{dvar_p\left(\widehat{m}\right)}{var_p\left(\widehat{m}\right)} = -\frac{\widehat{\tau}_s\left[1+(n-1)\rho\right]}{n\tau_\varepsilon + \widehat{\tau}_s\left[1+(n-1)\rho\right]}\frac{d\tau_\varepsilon}{\tau_\varepsilon} = \frac{1+(n-1)\rho}{n\tau_\varepsilon var_p\left(\widehat{m}\right)}\frac{d\tau_\varepsilon}{\tau_\varepsilon}. \tag{62}$$

These show that an increase in τ_ε will raise the value of ψ but lower $var_p\left(\widehat{m}\right)$.

Part (c) The overall effect on $\widetilde{\sigma}^2$ is determined by

$$\frac{\tau_\varepsilon}{\widetilde{\sigma}^2}\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} = 2\frac{\tau_\varepsilon}{\psi}\frac{d\psi}{d\tau_\varepsilon} + \frac{\tau_\varepsilon}{var_p\left(\widehat{m}\right)}\frac{dvar_p\left(\widehat{m}\right)}{d\tau_\varepsilon}.$$

Using (61) and (62), it can be shown that

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} \gtrless 0 \quad \Leftrightarrow \quad 2var_p\left(\widehat{m}\right) \gtrless var\left(\widehat{m}\right).$$

The condition on the right side is equivalent to

$$\frac{2\tau_s}{n\tau_\varepsilon + \tau_s\left[1+(n-1)\rho\right]} \gtrless \frac{\widehat{\tau}_s}{n\tau_\varepsilon + \widehat{\tau}_s\left[1+(n-1)\rho\right]},$$

which can be simplified to become

$$\tau_s \gtrless \frac{n\tau_\varepsilon\widehat{\tau}_s}{2n\tau_\varepsilon + \widehat{\tau}_s\left[1+(n-1)\rho\right]}.$$

This establishes the condition in part (c).

Part (d) Holding $\{\tau_s, \widehat{\tau}_s, \tau_\varepsilon, n\}$ constant, consider the total derivatives of ψ and $var_p\left(\widehat{m}\right)$ with respect to ρ, i.e.,

$$\frac{d\psi}{\psi} = -\frac{\tau_s(n-1)\rho}{n\tau_\varepsilon + \tau_s\left[1+(n-1)\rho\right]}\frac{d\rho}{\rho},$$

$$\frac{dvar_p\left(\widehat{m}\right)}{var_p\left(\widehat{m}\right)} = \frac{\widehat{\tau}_s(n-1)\rho}{n\tau_\varepsilon + \widehat{\tau}_s\left[1+(n-1)\rho\right]}\frac{d\rho}{\rho}.$$

Note that these equations are essentially the same as (61) and (62) but with opposite sides. The desired result can be obtained by using the same steps as in part (a). This completes the proof of Proposition 3.

Proof of Lemma 4

Since each b_i is independently drawn from the distribution $N\left(0, \tau_{b_i}^{-1}\right)$, $\mathbf{\Sigma}_b$ is a diagonal matrix with diagonal elements $\left(\tau_{b_1}^{-1}, ..., \tau_{b_n}^{-1}\right)$. Since each b_i is also independent of the state variable s, it follows that $\omega_i \equiv Cov\left(s, b_i\right) = 0$ and $\lambda_i \equiv Cov\left(s, m_i\right) = \sigma_s^2$ for all i, so that $\mathbf{\Lambda} = \sigma_s^2 \mathbf{1}_n$. It follows from (1) that

$$\mathbf{\Sigma}_m = \mathbf{\Sigma}_b + \mathbf{\Sigma}_\varepsilon + \sigma_s^2 \cdot \mathbf{1}_n \mathbf{1}_n^T.$$

The inverse of $\mathbf{\Sigma}_m$ can again be evaluated using the formula in (43). Since both $\mathbf{\Sigma}_b$ and $\mathbf{\Sigma}_\varepsilon$ are diagonal matrices, we can define

$$\mathbf{A} = \mathbf{\Sigma}_b + \mathbf{\Sigma}_\varepsilon = \begin{bmatrix} \tau_{b_1}^{-1} + \tau_{\varepsilon_1}^{-1} & 0 & \cdots & 0 \\ 0 & \tau_{b_2}^{-1} + \tau_{\varepsilon_2}^{-1} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \tau_{b_n}^{-1} + \tau_{\varepsilon_n}^{-1} \end{bmatrix},$$

and its inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} \widetilde{\tau}_1 & 0 & \cdots & 0 \\ 0 & \widetilde{\tau}_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \widetilde{\tau}_n \end{bmatrix},$$

where $\widetilde{\tau}_i \equiv \left(\tau_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1}\right)^{-1}$ for all i. From this point on, we can follow the same steps as in the proof of Lemma 2, with $\{\tau_{\varepsilon_1}, ..., \tau_{\varepsilon_n}\}$ replaced by $\{\widetilde{\tau}_1, ..., \widetilde{\tau}_n\}$. Hence,

$$E\left(s \mid \mathbf{m}\right) = \frac{\sum_{i=1}^n \widetilde{\tau}_i m_i}{\tau_s + \sum_{i=1}^n \widetilde{\tau}_i} = \underbrace{\frac{\sum_{i=1}^n \widetilde{\tau}_i}{\tau_s + \sum_{i=1}^n \widetilde{\tau}_i}}_{\psi} \cdot \underbrace{\sum_{i=1}^n \widetilde{\zeta}_i m_i}_{\widehat{m}},$$

where $\widetilde{\zeta}_i \equiv \widetilde{\tau}_i / \sum_{i=1}^n \widetilde{\tau}_i$ for all i, and

$$var\left(s \mid \mathbf{m}\right) = \frac{1}{\tau_s + \sum_{i=1}^n \widetilde{\tau}_i}.$$

Under the parties' beliefs, the covariance structure of $\{m_1, ..., m_n\}$ is given by

$$Cov_p\left(m_i, m_j\right) = \begin{cases} \widehat{\tau}_s^{-1} + \widehat{\tau}_{b_j}^{-1} + \tau_{\varepsilon_j}^{-1} & \text{for } i = j, \\ \widehat{\tau}_s^{-1} & \text{for } i \neq j, \end{cases}$$

for any given j. The perceived uncertainty is then given by

$$\widetilde{\sigma}^2 = \left(\frac{\sum_{i=1}^n \widetilde{\tau}_i}{\tau_s + \sum_{i=1}^n \widetilde{\tau}_i} \right)^2 var_p\left(\widehat{m}\right),$$

where

$$\begin{aligned} var_p\left(\widehat{m}\right) &= \sum_{j=1}^n \widetilde{\zeta}_j \sum_{i=1}^n \widetilde{\zeta}_i Cov_p\left(m_i, m_j\right) = \sum_{j=1}^n \widetilde{\zeta}_j \left[\widehat{\tau}_s^{-1} \sum_{i=1}^n \widetilde{\zeta}_i + \widetilde{\zeta}_j \left(\widehat{\tau}_{b_j}^{-1} + \tau_{\varepsilon_j}^{-1} \right) \right] \\ &= \widehat{\tau}_s^{-1} \left(\sum_{j=1}^n \widetilde{\zeta}_j \right)^2 + \sum_{j=1}^n \widetilde{\zeta}_j^2 \left(\widehat{\tau}_{b_j}^{-1} + \tau_{\varepsilon_j}^{-1} \right). \end{aligned}$$

Since $\sum_{j=1}^n \widetilde{\zeta}_j = 1$, we can simplify the above expression to become

$$var_p\left(\widehat{m}\right) = \widehat{\tau}_s^{-1} + \sum_{j=1}^n \left(\frac{\widetilde{\tau}_j}{\sum_{i=1}^n \widetilde{\tau}_i} \right)^2 \left(\widehat{\tau}_{b_j}^{-1} + \tau_{\varepsilon_j}^{-1} \right) = \frac{\left(\sum_{i=1}^n \widetilde{\tau}_i\right)^2 + \widehat{\tau}_s \sum_{j=1}^n \widetilde{\tau}_j^2 \left(\widehat{\tau}_{b_j}^{-1} + \tau_{\varepsilon_j}^{-1} \right)}{\widehat{\tau}_s \left(\sum_{i=1}^n \widetilde{\tau}_i\right)^2}. \quad (63)$$

Using this line of argument, we can get

$$var\left(\widehat{m}\right) = \frac{\left(\sum_{i=1}^n \widetilde{\tau}_i\right)^2 + \tau_s \sum_{j=1}^n \widetilde{\tau}_j^2 \left(\tau_{b_j}^{-1} + \tau_{\varepsilon_j}^{-1} \right)}{\tau_s \left(\sum_{i=1}^n \widetilde{\tau}_i\right)^2}. \quad (64)$$

This proves the results in Lemma 4.

Proof of Proposition 4

Part (a) Recall the definition of ψ in this case, which is repeated below

$$\psi = \frac{\sum_{i=1}^n \widetilde{\tau}_i}{\tau_s + \sum_{i=1}^n \widetilde{\tau}_i} > 0.$$

Holding τ_s and $\widetilde{\tau}_j$ for any $j \neq i$ constant, the total derivative of $\ln \psi$ is given by

$$d \ln \psi = \frac{d\psi}{\psi} = \frac{\tau_s \widetilde{\tau}_i}{\left(\sum_{i=1}^n \widetilde{\tau}_i\right)\left(\tau_s + \sum_{i=1}^n \widetilde{\tau}_i\right)} \frac{d\widetilde{\tau}_i}{\widetilde{\tau}_i}. \quad (65)$$

Next, recall the definition of $\widetilde{\tau}_i$, which is $\widetilde{\tau}_i \equiv \left(\tau_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1} \right)^{-1}$. It follows that

$$\frac{d\widetilde{\tau}_i}{\widetilde{\tau}_i} = \frac{\tau_{b_i}^{-1}}{\tau_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1}} \frac{d\tau_{b_i}}{\tau_{b_i}} + \frac{\tau_{\varepsilon_i}^{-1}}{\tau_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1}} \frac{d\tau_{\varepsilon_i}}{\tau_{\varepsilon_i}}. \quad (66)$$

Equations (65) and (66) together show that any increase in either τ_{b_i} or τ_{ε_i} will raise the value of ψ.

Part (b) Recall the definition of $var_p(\widehat{m})$ in (63). Since each $\widetilde{\tau}_i \equiv \left(\tau_{b_i}^{-1} + \tau_{\varepsilon_i}^{-1}\right)^{-1}$ is independent of $\widehat{\tau}_{b_i}$, it is obvious that $var_p(\widehat{m})$ is negatively related to $\widehat{\tau}_{b_i}$. Since ψ is also independent of $\widehat{\tau}_{b_i}$, it follows that any increase in $\widehat{\tau}_{b_i}$ will lower $\widetilde{\sigma}^2$ by lowering $var_p(\widehat{m})$.

Part (c) Suppose $\tau_{b_i} = \tau_b$, $\widehat{\tau}_{b_i} = \widehat{\tau}_b$ and $\tau_{\varepsilon_i} = \tau_\varepsilon$ for all i. Then ψ and $var_p(\widehat{m})$ can be simplified to become

$$\psi = \frac{n\widetilde{\tau}}{\tau_s + n\widetilde{\tau}} \qquad \text{and} \qquad var_p(\widehat{m}) = \frac{n + \widehat{\tau}_s\left(\widehat{\tau}_b^{-1} + \tau_\varepsilon^{-1}\right)}{n\widehat{\tau}_s}, \tag{67}$$

where $\widetilde{\tau} \equiv \left(\tau_b^{-1} + \tau_\varepsilon^{-1}\right)^{-1}$. This clearly shows that $var_p(\widehat{m})$ is independent of τ_b and inversely related to τ_ε. Similarly, (64) can be simplified to become

$$var(\widehat{m}) = \frac{n + \tau_s\left(\tau_b^{-1} + \tau_\varepsilon^{-1}\right)}{n\tau_s}. \tag{68}$$

Part (d) Suppose τ_b and $\widehat{\tau}_b$ are unchanged. Then from (67), we can get

$$\frac{d\psi}{\psi} = \frac{\tau_s}{\tau_s + n\widetilde{\tau}}\frac{d\widetilde{\tau}}{\widetilde{\tau}} = \frac{\tau_s\left(\tau_b^{-1} + \tau_\varepsilon^{-1}\right)}{n + \tau_s\left(\tau_b^{-1} + \tau_\varepsilon^{-1}\right)}\frac{d\widetilde{\tau}}{\widetilde{\tau}}, \tag{69}$$

where

$$\frac{d\widetilde{\tau}}{\widetilde{\tau}} = \frac{\tau_\varepsilon^{-1}}{\tau_b^{-1} + \tau_\varepsilon^{-1}}\frac{d\tau_\varepsilon}{\tau_\varepsilon}, \tag{70}$$

and

$$\frac{dvar_p(\widehat{m})}{var_p(\widehat{m})} = \frac{-\widehat{\tau}_s\tau_\varepsilon^{-1}}{n + \widehat{\tau}_s\left(\widehat{\tau}_b^{-1} + \tau_\varepsilon^{-1}\right)}\frac{d\tau_\varepsilon}{\tau_\varepsilon}. \tag{71}$$

The overall effect of changing τ_ε on $\widetilde{\sigma}^2$ is determined by

$$\frac{d\widetilde{\sigma}^2}{\widetilde{\sigma}^2} = 2\frac{d\psi}{\psi} + \frac{dvar_p(\widehat{m})}{var_p(\widehat{m})} = \left[\frac{2\tau_s}{n + \tau_s\left(\tau_b^{-1} + \tau_\varepsilon^{-1}\right)} - \frac{\widehat{\tau}_s}{n + \widehat{\tau}_s\left(\widehat{\tau}_b^{-1} + \tau_\varepsilon^{-1}\right)}\right]\tau_\varepsilon^{-1}\frac{d\tau_\varepsilon}{\tau_\varepsilon}.$$

The second equality is obtained by combining (69)-(71). This implies that

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} \gtrless 0 \qquad \text{if and only if} \qquad \frac{2\tau_s}{n + \tau_s\left(\tau_b^{-1} + \tau_\varepsilon^{-1}\right)} \gtrless \frac{\widehat{\tau}_s}{n + \widehat{\tau}_s\left(\widehat{\tau}_b^{-1} + \tau_\varepsilon^{-1}\right)},$$

which is equivalent to

$$\tau_s \gtrless \frac{n\widehat{\tau}_s}{2n + \widehat{\tau}_s\left(2\widehat{\tau}_b^{-1} + \tau_\varepsilon^{-1} - \tau_b^{-1}\right)}.$$

Using (67) and (68), we can show that this is equivalent to $2var_p\left(\widehat{m}\right) \gtrless var\left(\widehat{m}\right)$. This completes the proof of Proposition 4.

Proof of Proposition 5

Part (a) Given that $Cov\left(s,m\right) = \sigma_s^2 + \rho_{s,b}\sigma_s\sigma_b$ and $var\left(m\right) = \sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2 + 2\rho_{s,b}\sigma_s\sigma_b$, we can write

$$\psi = \frac{Cov\left(s,m\right)}{var\left(m\right)} = \frac{\sigma_s^2 + \rho_{s,b}\sigma_s\sigma_b}{\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2 + 2\rho_{s,b}\sigma_s\sigma_b}. \tag{72}$$

Straightforward differentiation gives

$$\frac{d\psi}{d\rho_{s,b}} = \frac{\sigma_s\sigma_b\left[var\left(m\right) - 2Cov\left(s,m\right)\right]}{\left[var\left(m\right)\right]^2},$$

where

$$var\left(m\right) - 2Cov\left(s,m\right) = \sigma_b^2 + \sigma_\varepsilon^2 - \sigma_s^2.$$

Hence,

$$\frac{d\widetilde{\sigma}^2}{d\rho_{s,b}} = 2\psi\frac{d\psi}{d\rho_{s,b}} \cdot var_p\left(\widehat{m}\right) \gtrless 0 \qquad \text{iff} \qquad Cov\left(s,m\right)\left[var\left(m\right) - 2Cov\left(s,m\right)\right] \gtrless 0.$$

Part (b) Differentiating the expression in (72) with respect to τ_s gives

$$\frac{d\psi}{d\tau_s} = \frac{\left\{\left(2\sigma_s + \rho_{s,b}\sigma_b\right)var\left(m\right) - 2\sigma_s\left(\sigma_s + \rho_{s,b}\sigma_b\right)^2\right\}}{\left[var\left(m\right)\right]^2}\underbrace{\frac{d\sigma_s}{d\tau_s}}_{(-)}.$$

The expression inside the curly brackets can be simplified as follows

$$\begin{aligned}
& \left(2\sigma_s + \rho_{s,b}\sigma_b\right)var\left(m\right) - 2\sigma_s\left(\sigma_s + \rho_{s,b}\sigma_b\right)^2 \\
= \; & \left(2\sigma_s + \rho_{s,b}\sigma_b\right)\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2 + 2\rho_{s,b}\sigma_s\sigma_b\right) - 2\sigma_s\left(\sigma_s + \rho_{s,b}\sigma_b\right)^2 \\
= \; & 2\sigma_s\left(\sigma_b^2 + \sigma_\varepsilon^2\right) + \rho_{s,b}\sigma_b\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2\right).
\end{aligned}$$

Hence,

$$\frac{d\psi}{d\tau_s} = \frac{\sigma_b\left(\sigma_s^2+\sigma_b^2+\sigma_\varepsilon^2\right)}{\left[var\left(m\right)\right]^2}\left[\rho_{s,b}+\frac{2\sigma_s\left(\sigma_b^2+\sigma_\varepsilon^2\right)}{\sigma_b\left(\sigma_s^2+\sigma_b^2+\sigma_\varepsilon^2\right)}\right]\underbrace{\frac{d\sigma_s}{d\tau_s}}_{(-)}.$$

This, together with

$$\psi = \frac{Cov\left(s,m\right)}{var\left(m\right)} = \frac{\sigma_s\sigma_b}{var\left(m\right)}\left(\rho_{s,b}+\frac{\sigma_s}{\sigma_b}\right),$$

implies that

$$\frac{d\widetilde{\sigma}^2}{d\tau_s} = 2\psi var_p\left(\widehat{m}\right)\cdot\frac{d\psi}{d\tau_s} \gtreqless 0 \qquad \text{iff} \qquad \left(\rho_{s,b}+\frac{\sigma_s}{\sigma_b}\right)\left[\rho_{s,b}+\frac{2\left(\sigma_b^2+\sigma_\varepsilon^2\right)}{\left(\sigma_s^2+\sigma_b^2+\sigma_\varepsilon^2\right)}\frac{\sigma_s}{\sigma_b}\right] \lesseqgtr 0.$$

Part (c) Differentiating the expression in (72) with respect to τ_b gives

$$\frac{d\psi}{d\tau_b} = \frac{\sigma_s\left\{\rho_{s,b}var\left(m\right)-2\left(\sigma_s+\rho_{s,b}\sigma_b\right)\left(\sigma_b+\rho_{s,b}\sigma_s\right)\right\}}{\left[var\left(m\right)\right]^2}\underbrace{\frac{d\sigma_b}{d\tau_b}}_{(-)}.$$

The term inside the curly brackets can be simplified as follows:

$$\begin{aligned}
&\rho_{s,b}var\left(m\right)-2\left(\sigma_s+\rho_{s,b}\sigma_b\right)\left(\sigma_b+\rho_{s,b}\sigma_s\right)\\
=\;&\rho_{s,b}\left(\sigma_s^2+\sigma_b^2+\sigma_\varepsilon^2+2\rho_{s,b}\sigma_s\sigma_b\right)-2\left(\sigma_s+\rho_{s,b}\sigma_b\right)\left(\sigma_b+\rho_{s,b}\sigma_s\right)\\
=\;&-\left[\rho_{s,b}\left(\sigma_s^2+\sigma_b^2-\sigma_\varepsilon^2\right)+2\sigma_s\sigma_b\right].
\end{aligned}$$

Hence,

$$\frac{d\psi}{d\tau_b} = -\frac{-\sigma_s}{\left[var\left(m\right)\right]^2}\left[\rho_{s,b}\left(\sigma_s^2+\sigma_b^2-\sigma_\varepsilon^2\right)+2\sigma_s\sigma_b\right]\underbrace{\frac{d\sigma_b}{d\tau_b}}_{(-)}$$

It follows that

$$\frac{d\widetilde{\sigma}^2}{d\tau_b} = 2\psi var_p\left(\widehat{m}\right)\cdot\frac{d\psi}{d\tau_b} \gtreqless 0 \qquad \text{iff} \qquad \left(\rho_{s,b}+\frac{\sigma_s}{\sigma_b}\right)\left[\rho_{s,b}\left(\sigma_s^2+\sigma_b^2-\sigma_\varepsilon^2\right)+2\sigma_s\sigma_b\right] \gtreqless 0.$$

Part (d) This result follows immediately from the fact that ψ is independent of $\left\{\widehat{\tau}_s,\widehat{\tau}_b,\widehat{\rho}_{s,b}\right\}$.

Part (e) Since $Cov\left(s,m\right)$ is independent of τ_ε, straightforward differentiation yields

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} = \left[Cov\left(s,m\right)\right]^2\left\{\frac{1}{\left[var\left(m\right)\right]^2}\frac{dvar_p\left(m\right)}{d\tau_\varepsilon}-2\frac{var_p\left(m\right)}{\left[var\left(m\right)\right]^3}\frac{dvar_p\left(m\right)}{d\tau_\varepsilon}\right\}.$$

Note that

$$\frac{dvar\left(m\right)}{d\tau_{\varepsilon}}=\frac{dvar_{p}\left(m\right)}{d\tau_{\varepsilon}}=\frac{d\sigma_{\varepsilon}^{2}}{d\tau_{\varepsilon}}=-\sigma_{\varepsilon}^{4}<0.$$

Combining these two equations gives

$$\frac{d\widetilde{\sigma}^{2}}{d\tau_{\varepsilon}}=\left[\frac{Cov\left(s,m\right)}{var\left(m\right)}\right]^{2}\left[\frac{2var_{p}\left(m\right)}{var\left(m\right)}-1\right]\sigma_{\varepsilon}^{4}\gtrless 0 \qquad \text{iff} \qquad 2var_{p}\left(m\right)\gtrless var\left(m\right).$$

This completes the proof of Proposition 5.

Derivation of (33)

Consider an arbitrary voter with $\delta_v \in \mathbb{R}$. Conditional on $\mathbf{m}$, the voter's expected utility if R wins is

$$\begin{aligned}
& E\left[U\left(x_{eq}^{*};\delta_{v}\right)\mid\mathbf{m}\right] \\
= & -E\left[\left(\delta_{v}+\psi\widehat{m}-x_{eq}^{*}\right)^{2}\mid\mathbf{m}\right]-var\left(s\mid\mathbf{m}\right) \\
= & -\left\{\left(\delta_{v}-x_{eq}^{*}\right)^{2}+2\left(\delta_{v}-x_{eq}^{*}\right)\psi\widehat{m}+\left(\psi\widehat{m}\right)^{2}+var\left(s\mid\mathbf{m}\right)\right\}.
\end{aligned}$$

Similarly, the voter's expected utility if L wins is

$$E\left[U\left(-x_{eq}^{*};\delta_{v}\right)\mid\mathbf{m}\right]=-\left\{\left(\delta_{v}+x_{eq}^{*}\right)^{2}+2\left(\delta_{v}+x_{eq}^{*}\right)\psi\widehat{m}+\left(\psi\widehat{m}\right)^{2}+var\left(s\mid\mathbf{m}\right)\right\}.$$

Before $\mathbf{m}$ is realised, the voter's expected utility is

$$\begin{aligned}
E\left[U\left(x_{eq}^{*};\delta_{v}\right)\right] &= \int_{0}^{\infty}E\left[U\left(x_{eq}^{*};\delta_{v}\right)\mid\mathbf{m}\right]dG\left(\widehat{m}\right)+\int_{-\infty}^{0}E\left[U\left(-x_{eq}^{*};\delta_{v}\right)\mid\mathbf{m}\right]dG\left(\widehat{m}\right) \\
&= -\left[\frac{1}{2}\left(\delta_{v}-x_{eq}^{*}\right)^{2}+\frac{1}{2}\left(\delta_{v}+x_{eq}^{*}\right)^{2}+var\left(s\mid\mathbf{m}\right)\right] \\
&\quad -\left[2\left(\delta_{v}-x_{eq}^{*}\right)\psi\int_{0}^{\infty}\widehat{m}dG\left(\widehat{m}\right)+2\left(\delta_{v}+x_{eq}^{*}\right)\psi\widehat{m}\int_{-\infty}^{0}\widehat{m}dG\left(\widehat{m}\right)\right] \\
&\quad -\psi^{2}\int_{-\infty}^{\infty}\widehat{m}^{2}dG\left(\widehat{m}\right) \\
&= 4\psi x_{eq}^{*}\int_{0}^{\infty}\widehat{m}dG\left(\widehat{m}\right)-\left[\delta_{v}^{2}+\left(x_{eq}^{*}\right)^{2}+var\left(s\mid\mathbf{m}\right)+\psi^{2}var\left(\widehat{m}\right)\right]. \qquad (73)
\end{aligned}$$

The last line follows from the fact that $G\left(\cdot\right)$ is the CDF of a symmetric distribution around zero, hence

$$\int_{-\infty}^{\infty}\widehat{m}dG\left(\widehat{m}\right)=0 \qquad \text{and} \qquad \int_{-\infty}^{0}\widehat{m}dG\left(\widehat{m}\right)=-\int_{0}^{\infty}\widehat{m}dG\left(\widehat{m}\right).$$

Using the formula,

$$\int_0^{\infty} x^{2n+1} \exp\left(-Ax^2\right) dx = \frac{n!}{2A^{n+1}}, \qquad \text{for } A > 0 \text{ and } n = 0, 1, 2, ...,$$

we can get

$$\int_0^{\infty} \widehat{m} dG\left(\widehat{m}\right) = \frac{1}{\sqrt{2\pi}\sigma_m} \int_0^{\infty} \widehat{m} \exp\left[-\left(2\sigma_m^2\right)^{-1}\left(\widehat{m}\right)^2\right] d\widehat{m} = \frac{\sigma_m}{\sqrt{2\pi}}.$$

Substituting this into (73) gives

$$E\left[U\left(x_{eq}^*; \delta_v\right)\right] = 2\sqrt{\frac{2}{\pi}} x_{eq}^* \psi \sigma_m - \left(x_{eq}^*\right)^2 - \left[\delta_v^2 + var\left(s \mid \mathbf{m}\right) + \psi^2 var\left(\widehat{m}\right)\right]. \tag{74}$$

Finally, by the law of total variance, we can get

$$var\left(s \mid \mathbf{m}\right) + \psi^2 var\left(\widehat{m}\right) = \tau_s^{-1}. \tag{75}$$

To see this, first recall that μ_s in the voter's subjective prior belief is normalised to zero, hence the variance of s in their prior belief is given by

$$\tau_s^{-1} = var\left(s\right) = E\left(s^2\right) = E\left[E\left(s^2 \mid \mathbf{m}\right)\right],$$

where the outer expectation is taken with respect to the joint distribution of $\mathbf{m}$. It follows that

$$\begin{aligned} \tau_s^{-1} &= E\left\{var\left(s \mid \mathbf{m}\right) + \left[E\left(s \mid \mathbf{m}\right)\right]^2\right\} \\ &= var\left(s \mid \mathbf{m}\right) + \int_{-\infty}^{\infty} \left[E\left(s \mid \mathbf{m}\right)\right]^2 dG\left(\widehat{m}\right). \end{aligned} \tag{76}$$

The last line uses the facts that $var\left(s \mid \mathbf{m}\right)$ is a deterministic constant according to (3) and $E\left(s \mid \mathbf{m}\right)$ is a function of $\widehat{m}$. Since the expected value of $\widehat{m}$ is zero,

$$\int_{-\infty}^{\infty} \left[E\left(s \mid \mathbf{m}\right)\right]^2 dG\left(\widehat{m}\right) = var\left[E\left(s \mid \mathbf{m}\right)\right] = \psi^2 var\left(\widehat{m}\right). \tag{77}$$

Substituting (77) into (76) gives (75). Combining (74) and (75) gives

$$E\left[U\left(x_{eq}^*; \delta_v\right)\right] = 2\sqrt{\frac{2}{\pi}} x_{eq}^* \psi \sigma_m - \left(x_{eq}^*\right)^2 - \left(\delta_v^2 + \tau_s^{-1}\right),$$

which is equation (33).

Proof of Proposition 6

The second inequality in (36) requires

$$2\sqrt{\frac{2}{\pi}}\psi\sigma_m \geq x^*_{eq} = \frac{2\phi - \gamma h(0)}{4h(0)\phi + 2}$$

$$\Leftrightarrow 2\sqrt{2}\psi\sigma_m [4h(0)\phi + 2] \geq 2\sqrt{\pi}\phi - \sqrt{\pi}\gamma h(0)$$

$$\Leftrightarrow 4\sqrt{2}\psi\sigma_m + \sqrt{\pi}\gamma h(0) \geq 2\left[\sqrt{\pi} - 4\sqrt{2}\psi\sigma_m h(0)\right]\phi.$$

There are two possible cases: If $\sqrt{\pi} - 4\sqrt{2}\psi\sigma_m h(0) \leq 0$, or equivalently,

$$\frac{\widetilde{\sigma}}{\psi\sigma_m} = \sqrt{\frac{var_p(\widehat{m})}{var(\widehat{m})}} \leq \frac{4}{\pi},$$

then the second inequality in (36) is automatically satisfied. This means $E\left[U\left(x^*_{eq};\delta_v\right)\right] \geq E\left[U(0;\delta_v)\right]$ for any $x^*_{eq} \geq 0$.

But if $\sqrt{\pi} - 4\sqrt{2}\psi\sigma_m h(0) > 0$, or equivalently,

$$\frac{\widetilde{\sigma}}{\psi\sigma_m} = \sqrt{\frac{var_p(\widehat{m})}{var(\widehat{m})}} > \frac{4}{\pi},$$

then the second inequality in (36) holds if and only if

$$\phi \leq \frac{\sqrt{2}\psi\sigma_m + \sqrt{\pi}\gamma h(0)}{2\left[\sqrt{\pi} - 4\sqrt{2}\psi\sigma_m h(0)\right]} = \frac{\sqrt{\pi}\left(8\psi\sigma_m \cdot \widetilde{\sigma} + \gamma\right)}{2\sqrt{2}\left[\pi\widetilde{\sigma} - 4\psi\sigma_m\right]}.$$

This completes the proof of Proposition 6.

Proof of Proposition 7

Suppose $var(\widehat{m}) = var_p(\widehat{m})$, which implies $\psi\sigma_m = \widetilde{\sigma}$. Then equation (33) can be rewritten as

$$E\left[U\left(x^*_{eq};\delta_v\right)\right] = 2\sqrt{\frac{2}{\pi}}\widetilde{\sigma}x^*_{eq} - \left(x^*_{eq}\right)^2 - \left(\delta_v^2 + \tau_s^{-1}\right),$$

for any $x^*_{eq} > 0$, or equivalently $\widetilde{\sigma} > \sigma_{\min}$. Straightforward differentiation yields

$$\begin{aligned}\frac{d}{dz}E\left[U\left(x^*_{eq};\delta_v\right)\right] &= 2\sqrt{\frac{2}{\pi}}\left(\widetilde{\sigma}\frac{dx^*_{eq}}{dz} + x^*_{eq}\frac{d\widetilde{\sigma}}{dz}\right) - 2x^*_{eq}\cdot\frac{dx^*_{eq}}{dz} \\ &= \left[2\left(\sqrt{\frac{2}{\pi}}\widetilde{\sigma} - x^*_{eq}\right)\frac{dx^*_{eq}}{d\widetilde{\sigma}} + 2\sqrt{\frac{2}{\pi}}x^*_{eq}\right]\frac{d\widetilde{\sigma}}{dz}.\end{aligned}$$

The second line uses the chain rule of differentiation,

$$\frac{dx_{eq}^*}{dz} = \frac{dx_{eq}^*}{d\widetilde{\sigma}} \cdot \frac{d\widetilde{\sigma}}{dz}.$$

As shown in Corollary 1, x_{eq}^* is strictly increasing in $\widetilde{\sigma}$ whenever $\widetilde{\sigma} > \sigma_{\min}$. This, together with $x_{eq}^* > 0$ and (38), means that $\sqrt{2/\pi}\widetilde{\sigma} \geq x_{eq}^*$ is a sufficient condition for

$$\frac{d}{dz} E\left[U\left(x_{eq}^*; \delta_v\right)\right] > 0.$$

Recall that the extent of polarisation x_{eq}^* is determined by

$$x_{eq}^* = \frac{2\phi - \gamma h(0)}{4h(0)\phi + 2}, \qquad \text{where } h(0) \equiv 1/\left(\widetilde{\sigma}\sqrt{2\pi}\right).$$

Hence, x_{eq}^* can also be expressed as

$$x_{eq}^* = \frac{2\sqrt{2\pi}\phi\widetilde{\sigma} - \gamma}{4\phi + 2\sqrt{2\pi}\widetilde{\sigma}} = \frac{2\sqrt{2\pi}\phi\left(\widetilde{\sigma} - \sigma_{\min}\right)}{4\phi + 2\sqrt{2\pi}\widetilde{\sigma}}, \qquad \text{where } \sigma_{\min} \equiv \frac{\gamma}{2\sqrt{2\pi}\phi}.$$

The sufficient condition $\sqrt{2/\pi}\widetilde{\sigma} \geq x_{eq}^*$ can now be rewritten as

$$\widetilde{\sigma} \geq \frac{\pi\phi\left(\widetilde{\sigma} - \sigma_{\min}\right)}{2\phi + \sqrt{2\pi}\widetilde{\sigma}}$$

$$\Leftrightarrow \widetilde{\sigma}\left(2\phi + \sqrt{2\pi}\widetilde{\sigma}\right) \geq \pi\phi\left(\widetilde{\sigma} - \sigma_{\min}\right)$$

$$\Leftrightarrow \sqrt{2\pi}\widetilde{\sigma}^2 - \phi(\pi - 2)\widetilde{\sigma} + \pi\phi\sigma_{\min} \geq 0. \tag{78}$$

Consider the following quadratic equation:

$$\sqrt{2\pi}y^2 - \phi(\pi - 2)y + \pi\phi\sigma_{\min} = 0.$$

Since $\phi(\pi - 2) > 0$ and $\pi\phi\sigma_{\min} > 0$, this equation has two distinct real roots. The sum and the product of roots are, respectively, given by $-\phi(\pi - 2) < 0$ and $\pi\phi\sigma_{\min} > 0$. Hence, the two roots must be negative-valued. This in turn implies that $\sqrt{2\pi}y^2 - \phi(\pi - 2)y + \pi\phi\sigma_{\min} > 0$ for all $y \geq 0$. Hence, (78) is valid for any $\widetilde{\sigma} > \sigma_{\min} > 0$. This proves the desired result.

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