Mathematical Appendix

“Information Quality, Disagreement and Political Polarisation”

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This contains proofs of the theoretical results in the paper.

Proof of Lemma 1

The proof is based on a well-known result concerning conditional multivariate normal distributions which is stated as follows [see, for instance, Greene (2012, p.1042, Theorem B.7)]. Suppose $[\mathbf{X}_1, \mathbf{X}_2]$ has a joint multivariable normal distribution $\mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}.$$

The marginal distribution of $\mathbf{X}_i$ is given by $\mathbf{N}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_{ii})$ for $i \in \{1, 2\}$. Then the conditional distribution of $\mathbf{X}_1$ given $\mathbf{X}_2$ is normal with mean vector

$$\boldsymbol{\mu}_{1,2} = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2),$$

and covariance matrix

$$\boldsymbol{\Sigma}_{11,2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}.$$

In order to apply this result, first note that $(s, \mathbf{b}, \mathbf{m})$ has a joint multivariate normal distribution with mean vector $\boldsymbol{\mu}^\dagger$ and covariance matrix $\boldsymbol{\Sigma}^\dagger$ given by

$$\boldsymbol{\mu}^\dagger = \begin{bmatrix} \mu_s \\ \boldsymbol{\mu}_b \\ \boldsymbol{\mu}_m \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\Sigma}^\dagger = \begin{bmatrix} \sigma_s^2 & \boldsymbol{\Omega} & \boldsymbol{\Lambda} \\ \boldsymbol{\Omega}^T & \boldsymbol{\Sigma}_b & \boldsymbol{\Theta} \\ \boldsymbol{\Lambda}^T & \boldsymbol{\Theta}^T & \boldsymbol{\Sigma}_m \end{bmatrix}.$$

The meaning of $\boldsymbol{\Lambda}$ in the covariance matrix has been explained in the main text. The covariances between $\mathbf{b}$ and $\mathbf{m}$ are captured by the n-by-n matrix $\boldsymbol{\Theta} \equiv E\left[(\mathbf{b} - \boldsymbol{\mu}_\mathbf{b})(\mathbf{m} - \boldsymbol{\mu}_m)^T\right]$. The (i, j)th element of $\boldsymbol{\Theta}$ is denoted by $\theta_{i,j} \equiv Cov(b_i, m_j) = \omega_i + Cov(b_i, b_j)$.

Using the theorem mentioned above, the posterior distribution of $(s, \mathbf{b})$ after observing $\mathbf{m}$ is a normal distribution with mean vector

$$\boldsymbol{\mu}' = \begin{bmatrix} \mu_s \\ \boldsymbol{\mu}_b \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Lambda} \\ \boldsymbol{\Theta} \end{bmatrix} \boldsymbol{\Sigma}_m^{-1}(\mathbf{m} - \boldsymbol{\mu}_m), \tag{A1}$$

and covariance matrix

$$\mathbf{\Sigma}' = \begin{bmatrix} \sigma_s^2 & \mathbf{\Omega} \\ \mathbf{\Omega}^T & \mathbf{\Sigma}_b \end{bmatrix} - \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{\Theta} \end{bmatrix} \mathbf{\Sigma}_m^{-1} \begin{bmatrix} \mathbf{\Lambda}^T & \mathbf{\Theta}^T \end{bmatrix}. \tag{A2}$$

It follows that the marginal distribution of s in the voters' posterior belief is also normal. To derive the posterior mean and posterior variance of s, we first define $\kappa_{i,j}$ as the element on the ith row and jth column of $\mathbf{\Sigma}_m^{-1}$. Then

$$\begin{aligned}
\begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{\Theta} \end{bmatrix} \mathbf{\Sigma}_m^{-1} (\mathbf{m} - \boldsymbol{\mu}_m) &= \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{bmatrix} \begin{bmatrix} \kappa_{1,1} & \cdots & \kappa_{1,n} \\ \vdots & \ddots & \\ \kappa_{n,1} & \cdots & \kappa_{n,n} \end{bmatrix} \begin{bmatrix} m_1 - \mu_{m_1} \\ \vdots \\ m_n - \mu_{m_n} \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{bmatrix}}_{(n+1)\text{-by-}n} \underbrace{\begin{bmatrix} \sum_{j=1}^n \kappa_{1,j} \left(m_j - \mu_{m_j} \right) \\ \vdots \\ \sum_{j=1}^n \kappa_{n,j} \left(m_j - \mu_{m_j} \right) \end{bmatrix}}_{n\text{-by-}1}.
\end{aligned}$$

The first entry in the resulting $(n+1)$-by-1 vector is

$$\mathbf{\Lambda} \mathbf{\Sigma}_m^{-1} (\mathbf{m} - \boldsymbol{\mu}_m) = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \kappa_{i,j} \left(m_j - \mu_{m_j} \right).$$

It follows from (A1) that the posterior mean of s is

$$\begin{aligned}
E(s \mid \mathbf{m}) &= \mu_s + \sum_{i=1}^n \sum_{j=1}^n \lambda_i \kappa_{i,j} \left(m_j - \mu_{m_j} \right) \\
&= \mu_s + \sum_{j=1}^n \underbrace{\left(\sum_{i=1}^n \lambda_i \kappa_{i,j} \right)}_{\alpha_j} \left(m_j - \mu_{m_j} \right).
\end{aligned}$$

Similarly,

$$\begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{\Theta} \end{bmatrix} \mathbf{\Sigma}_m^{-1} \begin{bmatrix} \mathbf{\Lambda}^T & \mathbf{\Theta}^T \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{bmatrix} \begin{bmatrix} \kappa_{1,1} & \cdots & \kappa_{1,n} \\ \vdots & \ddots & \\ \kappa_{n,1} & \cdots & \kappa_{n,n} \end{bmatrix} \begin{bmatrix} \lambda_1 & \theta_{1,1} & \cdots & \theta_{n,1} \\ \vdots & & & \vdots \\ \lambda_n & \theta_{1,n} & \cdots & \theta_{n,n} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \\ \theta_{1,1} & \cdots & \theta_{1,n} \\ \vdots & & \vdots \\ \theta_{n,1} & \cdots & \theta_{n,n} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^n \kappa_{1,j}\lambda_j & \sum_{j=1}^n \kappa_{1,j}\theta_{1,j} & \cdots & \sum_{j=1}^n \kappa_{1,j}\theta_{n,j} \\ \vdots & & & \vdots \\ \sum_{j=1}^n \kappa_{n,j}\lambda_j & \sum_{j=1}^n \kappa_{n,j}\theta_{1,j} & \cdots & \sum_{j=1}^n \kappa_{n,j}\theta_{n,j} \end{bmatrix}.$$

The $(1,1)$th element of the resulting $(n+1)$-by-$(n+1)$ matrix is

$$\mathbf{\Lambda}\mathbf{\Sigma}_m^{-1}\mathbf{\Lambda}^T = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \kappa_{i,j} \lambda_j.$$

It follows from (A2) that the posterior variance of s is

$$var\,(s \mid \mathbf{m}) = \sigma_s^2 - \sum_{j=1}^n \left(\sum_{i=1}^n \lambda_i \kappa_{i,j} \right) \lambda_j.$$

This completes the proof of Lemma 1.

Proof of Lemma 2

Suppose each b_i, $i \in \{1, 2, ..., n\}$, is a deterministic constant normalised to zero, and suppose $\mu_s = 0$. Then $(s, \mathbf{m})$ has a joint multivariate normal distribution with zero mean vector and covariance matrix $\boldsymbol{V}$ given by

$$\boldsymbol{V} = \begin{bmatrix} \sigma_s^2 & \mathbf{\Lambda}^T \\ \mathbf{\Lambda} & \mathbf{\Sigma}_m \end{bmatrix},$$

where $\mathbf{\Lambda} = \sigma_s^2 \times \mathbf{1}_n$. Thus, for Case 1 and Case 2 where signals are unbiased, $\lambda_i = \sigma_s^2$ for all i.

Suppose each ε_i is drawn from the distribution $N\left(0, \sigma_{\varepsilon_i}^2\right)$, where $\sigma_{\varepsilon_i}^2 = \tau_{\varepsilon_i}^{-1}$. Then the covariance structure of $\{m_1, ..., m_n\}$ is given by

$$Cov\,(m_i, m_j) = \begin{cases} \sigma_s^2 + \sigma_{\varepsilon_i}^2 & \text{for } i = j, \\ \sigma_s^2 & \text{for } i \neq j. \end{cases}$$

Hence, $\mathbf{\Sigma}_m$ can be expressed as the sum of two n-by-n matrices,

$$\mathbf{\Sigma}_m = \mathbf{A} + \sigma_s^2 \mathbf{1}_n \mathbf{1}_n^T,$$

where $\mathbf{A}$ is a diagonal matrix with diagonal elements $\left(\sigma_{\varepsilon_1}^2, ..., \sigma_{\varepsilon_n}^2\right)$. The inverse of $\mathbf{\Sigma}_m$ can be derived using equation (3) in Henderson and Searle (1981, p.53). Specifically, this equation states that for any matrix $\mathbf{M} = \mathbf{A} + r\mathbf{u}\mathbf{v}^T$, where $\mathbf{A}$ can be any invertible matrix, r is a scalar, $\mathbf{u}$ is a column vector and $\mathbf{v}^T$ is a row vector, the inverse can be expressed as

$$\mathbf{M}^{-1} = \mathbf{A}^{-1} - \xi \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}, \tag{A3}$$

where

$$\xi = \frac{r}{1 + r\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}.$$

Hence, by setting $r = \sigma_s^2$, $\mathbf{u} = \mathbf{1}_n$ and $\mathbf{v}^T = \mathbf{1}_n^T$, we can get

$$\mathbf{\Sigma}_m^{-1} = \mathbf{A}^{-1} - \xi \mathbf{A}^{-1}\mathbf{1}_n\mathbf{1}_n^T\mathbf{A}^{-1}, \tag{A4}$$

where

$$\xi = \frac{\sigma_s^2}{1 + \sigma_s^2\mathbf{1}_n^T\mathbf{A}^{-1}\mathbf{1}_n}.$$

Since $\mathbf{A}$ is a diagonal matrix, its inverse is simply

$$\mathbf{A}^{-1} = \begin{bmatrix} \tau_{\varepsilon_1} & 0 & \cdots & 0 \\ 0 & \tau_{\varepsilon_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \tau_{\varepsilon_n} \end{bmatrix}. \tag{A5}$$

It follows that $\mathbf{1}_n^T\mathbf{A}^{-1}\mathbf{1}_n = \sum_{i=1}^n \tau_{\varepsilon_i}$, and

$$\xi = \frac{\sigma_s^2}{1 + \sigma_s^2\sum_{i=1}^n \tau_{\varepsilon_i}} = \frac{1}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}}, \tag{A6}$$

where $\tau_s \equiv \sigma_s^{-2}$. In addition,

$$\mathbf{A}^{-1}\mathbf{1}_n\mathbf{1}_n^T\mathbf{A}^{-1} = \begin{bmatrix} \tau_{\varepsilon_1} \\ \tau_{\varepsilon_2} \\ \vdots \\ \tau_{\varepsilon_n} \end{bmatrix} \begin{bmatrix} \tau_{\varepsilon_1} & \tau_{\varepsilon_2} & \cdots & \tau_{\varepsilon_n} \end{bmatrix} = \begin{bmatrix} \tau_{\varepsilon_1}^2 & \tau_{\varepsilon_1}\tau_{\varepsilon_2} & \cdots & \tau_{\varepsilon_1}\tau_{\varepsilon_n} \\ \tau_{\varepsilon_1}\tau_{\varepsilon_2} & \tau_{\varepsilon_2}^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \tau_{\varepsilon_1}\tau_{\varepsilon_n} & \cdots & \cdots & \tau_{\varepsilon_n}^2 \end{bmatrix}. \quad \text{(A7)}$$

Using (A4)-(A7), we can express the elements on any jth column of $\mathbf{\Sigma}_m^{-1}$ as

$$\kappa_{i,j} = \begin{cases} \tau_{\varepsilon_j} - \xi {\tau_{\varepsilon_j}}^2 & \text{for } i = j, \\ -\xi\tau_{\varepsilon_i}\tau_{\varepsilon_j} & \text{for } i \neq j, \end{cases}$$

$$\Rightarrow \alpha_j = \sum_{i=1}^n \lambda_i\kappa_{i,j} = \sigma_s^2\tau_{\varepsilon_j}\left(1 - \xi\sum_{i=1}^n \tau_{\varepsilon_i}\right) = \frac{\tau_{\varepsilon_j}}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}}.$$

Substituting these and $\lambda_i = \sigma_s^2$ into Equations (2) and (3) in the paper gives

$$E\left(s \mid \mathbf{m}\right) = \sum_{i=1}^n \alpha_i m_i = \frac{\sum_{i=1}^n \tau_{\varepsilon_i} m_i}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}} = \underbrace{\frac{\sum_{i=1}^n \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}}}_{\psi} \cdot \underbrace{\sum_{i=1}^n \zeta_i m_i}_{\widehat{m}},$$

where $\zeta_i \equiv \tau_{\varepsilon_i}/\sum_{i=1}^n \tau_{\varepsilon_i}$ for all i, and

$$var\left(s \mid \mathbf{m}\right) = \frac{1}{\tau_s} - \sum_{i=1}^n \lambda_i\alpha_i = \frac{1}{\tau_s} - \frac{1}{\tau_s}\frac{\sum_{i=1}^n \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}} = \frac{1}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}}.$$

Finally, under the parties' beliefs, the covariance structure of $\{m_1, ..., m_n\}$ is given by

$$Cov_p\left(m_i, m_j\right) = \begin{cases} \widehat{\sigma}_s^2 + \sigma_{\varepsilon_j}^2 & \text{for } i = j, \\ \widehat{\sigma}_s^2 & \text{for } i \neq j, \end{cases}$$

for any given j, and the perceived uncertainty is given by

$$\widetilde{\sigma}^2 = \left(\frac{\sum_{i=1}^n \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^n \tau_{\varepsilon_i}}\right)^2 var_p\left(\widehat{m}\right),$$

where

$$var_p\left(\widehat{m}\right) = \sum_{j=1}^n \zeta_j \sum_{i=1}^n \zeta_i Cov_p\left(m_i, m_j\right) = \sum_{j=1}^n \zeta_j\left(\widehat{\sigma}_s^2\sum_{i=1}^n \zeta_i + \zeta_j\sigma_{\varepsilon_j}^2\right).$$

Since $\sum_{j=1}^n \zeta_j = 1$ and $\zeta_j\sigma_{\varepsilon_j}^2 = \left(\sum_{i=1}^n \tau_{\varepsilon_i}\right)^{-1}$ for all j, we can simplify the above expression to become

$$var_p\left(\widehat{m}\right) = \widehat{\sigma}_s^2 + \left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)^{-1} = \frac{\left(\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)}{\widehat{\tau}_s \left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)}. \tag{A8}$$

Using the same line of argument and replacing $\widehat{\sigma}_s^2$ with σ_s^2, we can show that

$$var\left(\widehat{m}\right) = \frac{\left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)}{\tau_s \left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)}, \tag{A9}$$

which is the unconditional variance of $\widehat{m}$ under the voters' subjective prior belief. This completes the proof of Lemma 2.

Proof of Proposition 2

Recall that perceived uncertainty $\widetilde{\sigma}^2$ can be expressed as

$$\widetilde{\sigma}^2 = \underbrace{\left(\frac{\sum_{i=1}^{n} \tau_{\varepsilon_i}}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}}\right)^2}_{\psi^2} \cdot \underbrace{\frac{\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}}{\widehat{\tau}_s \sum_{i=1}^{n} \tau_{\varepsilon_i}}}_{var_p(\widehat{m})}.$$

It is clear that any changes in τ_s will only affect ψ but not $var_p\left(\widehat{m}\right)$. Likewise, any changes in $\widehat{\tau}_s$ will only affect $var_p\left(\widehat{m}\right)$ but not ψ. Consider the logarithm of ψ,

$$\ln \psi = \ln\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right) - \ln\left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right).$$

Totally differentiating this with respect to $\{\psi, \tau_s, \tau_{\varepsilon_i}\}$ gives

$$\frac{d\psi}{\psi} = -\frac{\tau_s}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} \frac{d\tau_s}{\tau_s} + \frac{\tau_s \tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)\left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)} \frac{d\tau_{\varepsilon_i}}{\tau_{\varepsilon_i}}.$$

Suppose $d\tau_{\varepsilon_i} = 0$, then we have

$$\frac{d\psi}{d\tau_s} = -\frac{\psi}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} < 0 \qquad \Rightarrow \qquad \frac{d\widetilde{\sigma}^2}{d\tau_s} < 0. \tag{A10}$$

On the other hand, if $d\tau_s = 0$, then

$$\frac{d\psi}{d\tau_{\varepsilon_i}} = \frac{\tau_s \psi}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)\left(\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)} > 0, \tag{A11}$$

$$\Rightarrow \frac{\tau_{\varepsilon_i}}{\psi} \frac{d\psi}{d\tau_{\varepsilon_i}} = \frac{\tau_{\varepsilon_i}}{\sum_{i=1}^{n} \tau_{\varepsilon_i}} \frac{\tau_s}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} = \frac{\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)^2} \frac{1}{var\left(\widehat{m}\right)}.$$

The second equality follows from (A9). Similarly, totally differentiating $\ln[var_p(\widehat{m})]$ with respect to $\{\psi, \widehat{\tau}_s, \tau_{\varepsilon_i}\}$ gives

$$\ln[var_p(\widehat{m})] = \ln\left[\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right] - \ln\widehat{\tau}_s - \ln\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)$$

$$\frac{dvar_p(\widehat{m})}{var_p(\widehat{m})} = -\frac{\sum_{i=1}^{n} \tau_{\varepsilon_i}}{\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} \frac{d\widehat{\tau}_s}{\widehat{\tau}_s} - \frac{\widehat{\tau}_s \tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)\left(\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)} \frac{d\tau_{\varepsilon_i}}{\tau_{\varepsilon_i}}.$$

When all other factors except $\widehat{\tau}_s$ are kept constant,

$$\frac{dvar_p(\widehat{m})}{d\widehat{\tau}_s} = -\frac{\sum_{i=1}^{n} \tau_{\varepsilon_i}}{\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} \frac{var_p(\widehat{m})}{\widehat{\tau}_s} < 0 \quad \Rightarrow \quad \frac{d\widetilde{\sigma}^2}{d\widehat{\tau}_s} < 0. \tag{A12}$$

If $d\tau_s = 0$, then

$$\frac{dvar_p(\widehat{m})}{d\tau_{\varepsilon_i}} = -\frac{\widehat{\tau}_s var_p(\widehat{m})}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)\left(\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)} < 0, \tag{A13}$$

$$\Rightarrow \frac{\tau_{\varepsilon_i}}{var_p(\widehat{m})} \frac{dvar_p(\widehat{m})}{d\tau_{\varepsilon_i}} = -\frac{\tau_{\varepsilon_i}}{\sum_{i=1}^{n} \tau_{\varepsilon_i}} \frac{\widehat{\tau}_s}{\left(\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}\right)} = -\frac{\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)^2} \frac{1}{var_p(\widehat{m})}.$$

The second equality follows from (A8). Equations (A11) and (A13) together prove the statement in part (b).

Holding τ_s and $\widehat{\tau}_s$ constant, the overall effect of changing τ_{ε_i} on $\widetilde{\sigma}^2$ can be determined by

$$\begin{aligned} \frac{\tau_{\varepsilon_i}}{\widetilde{\sigma}^2} \frac{d\widetilde{\sigma}^2}{d\tau_{\varepsilon_i}} &= 2\frac{\tau_{\varepsilon_i}}{\psi} \frac{d\psi}{d\tau_{\varepsilon_i}} + \frac{\tau_{\varepsilon_i}}{var_p(\widehat{m})} \frac{dvar_p(\widehat{m})}{d\tau_{\varepsilon_i}} \\ &= \left[\frac{2}{var(\widehat{m})} - \frac{1}{var_p(\widehat{m})}\right] \frac{\tau_{\varepsilon_i}}{\left(\sum_{i=1}^{n} \tau_{\varepsilon_i}\right)^2}. \end{aligned}$$

Hence,

$$\frac{d\widetilde{\sigma}^2}{d\tau_{\varepsilon_i}} \gtrless 0 \Leftrightarrow 2var_p(\widehat{m}) \gtrless var(\widehat{m}).$$

Using (A8) and (A9), we can show that

$$2var_p(\widehat{m}) \gtrless var(\widehat{m}) \quad \text{if and only if} \quad \frac{2\tau_s}{\tau_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}} \gtrless \frac{\widehat{\tau}_s}{\widehat{\tau}_s + \sum_{i=1}^{n} \tau_{\varepsilon_i}},$$

which is equivalent to

$$\tau_s \gtrless \frac{\widehat{\tau}_s \sum_{i=1}^{n} \tau_{\varepsilon_i}}{\widehat{\tau}_s + 2\sum_{i=1}^{n} \tau_{\varepsilon_i}}.$$

This completes the proof of Proposition 2.

Proof of Lemma 3

Suppose $\rho \geq -1/(n-1)$. The inverse of $\boldsymbol{\Sigma}_\varepsilon$ can be shown to take the following form

$$\boldsymbol{\Sigma}_\varepsilon^{-1} = \frac{\tau_\varepsilon}{1+(n-2)\rho-(n-1)\rho^2} \begin{bmatrix} 1+(n-2)\rho & -\rho & \cdots & -\rho \\ -\rho & 1+(n-2)\rho & \cdots & -\rho \\ \vdots & & \ddots & \vdots \\ -\rho & \cdots & -\rho & 1+(n-2)\rho \end{bmatrix}. \quad \text{(A14)}$$

To see this, note that all diagonal entries of $\boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Sigma}_\varepsilon^{-1}$ are given by

$$\frac{1}{1+(n-2)\rho-(n-1)\rho^2}\left[1+(n-2)\rho-(n-1)\rho^2\right] = 1,$$

and all off-diagonal elements of $\boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Sigma}_\varepsilon^{-1}$ are given by

$$\frac{1}{1+(n-2)\rho-(n-1)\rho^2}\left\{-\rho+[1+(n-2)\rho]\rho+(n-2)\rho^2\right\} = 0.$$

Define the notation ν according to

$$\nu \equiv \frac{\tau_\varepsilon}{1+(n-2)\rho-(n-1)\rho^2} = \frac{\tau_\varepsilon}{(1-\rho)[1+(n-1)\rho]}.$$

The covariances among the signals $\{m_1, ..., m_n\}$ are given by $Cov(m_i, m_j) = \sigma_s^2 + Cov(\varepsilon_i, \varepsilon_j)$, which implies

$$\boldsymbol{\Sigma}_m = \boldsymbol{\Sigma}_\varepsilon + +\sigma_s^2 \mathbf{1}_n \mathbf{1}_n^T.$$

Using the same formula in (A3), we can get

$$\boldsymbol{\Sigma}_m^{-1} = \boldsymbol{\Sigma}_\varepsilon^{-1} - \xi \boldsymbol{\Sigma}_\varepsilon^{-1} \mathbf{1}_n \mathbf{1}_n^T \boldsymbol{\Sigma}_\varepsilon^{-1}, \quad \text{(A15)}$$

where

$$\xi = \frac{\sigma_s^2}{1+\sigma_s^2 \mathbf{1}_n^T \boldsymbol{\Sigma}_\varepsilon^{-1} \mathbf{1}_n} = \frac{1}{\tau_s + \mathbf{1}_n^T \boldsymbol{\Sigma}_\varepsilon^{-1} \mathbf{1}_n}.$$

It is straightforward to show that

$$\mathbf{1}_n^T \boldsymbol{\Sigma}_\varepsilon^{-1} \mathbf{1}_n = n\nu(1-\rho) = \frac{n\tau_\varepsilon}{1+(n-1)\rho}.$$

$$\Rightarrow \xi = \frac{1+(n-1)\rho}{n\tau_\varepsilon + \tau_s\left[1+(n-1)\rho\right]} \tag{A16}$$

On the other hand,

$$\boldsymbol{\Sigma}_\varepsilon^{-1}\mathbf{1}_n\mathbf{1}_n^T\boldsymbol{\Sigma}_\varepsilon^{-1} = \nu^2(1-\rho)^2\,\mathbf{1}_n\mathbf{1}_n^T. \tag{A17}$$

Using (A15)-(A17), we can write the elements on any jth column of $\boldsymbol{\Sigma}_m^{-1}$ as

$$\kappa_{i,j} = \begin{cases} \nu\left[1+(n-2)\rho\right] - \xi\nu^2(1-\rho)^2 & \text{for } i=j, \\ -\nu\rho - \xi\nu^2(1-\rho)^2 & \text{for } i\neq j. \end{cases}$$

Using these and $\lambda_i = \sigma_s^2 = \tau_s^{-1}$, we can get

$$\begin{aligned} \alpha_j &= \sum_{i=1}^{n}\lambda_i\kappa_{i,j} = \frac{1}{\tau_s}\left[\nu(1-\rho) - n\xi\nu^2(1-\rho)^2\right] \\ &= \frac{\tau_\varepsilon}{\tau_s\left[1+(n-1)\rho\right]}\left[1 - \frac{n\tau_\varepsilon\xi}{1+(n-1)\rho}\right] \\ &= \frac{\tau_\varepsilon}{n\tau_\varepsilon+\tau_s\left[1+(n-1)\rho\right]}. \end{aligned}$$

Hence, the posterior mean and posterior variance of s are given by

$$E(s\mid\mathbf{m}) = \underbrace{\frac{n\tau_\varepsilon}{n\tau_\varepsilon+\tau_s\left[1+(n-1)\rho\right]}}_{\psi}\cdot\underbrace{\frac{1}{n}\sum_{i=1}^{n}m_i}_{\widehat{m}},$$

$$var(s\mid\mathbf{m}) = \frac{1}{\tau_s}\left(1-\sum_{i=1}^{n}\alpha_j\right) = \frac{1+(n-1)\rho}{n\tau_\varepsilon+\tau_s\left[1+(n-1)\rho\right]}.$$

From the parties' perspective, the covariance structure of $\{m_1, ..., m_n\}$ is now given by

$$Cov_p(m_i, m_j) = \begin{cases} \widehat{\tau}_s^{-1} + \tau_\varepsilon^{-1} & \text{for } i=j, \\ \widehat{\tau}_s^{-1} + \tau_\varepsilon^{-1}\rho & \text{for } i\neq j, \end{cases} \tag{A18}$$

and the perceived uncertainty is given by

$$\widetilde{\sigma}^2 = \left\{\frac{n\tau_\varepsilon}{n\tau_\varepsilon+\tau_s\left[1+(n-1)\rho\right]}\right\}^2 var_p(\widehat{m}),$$

where

$$
\begin{aligned}
var_p(\widehat{m}) &= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} Cov_p(m_i, m_j) \\
&= \frac{1}{n^2}\sum_{j=1}^{n}\left\{n\widehat{\tau}_s^{-1} + \tau_\varepsilon^{-1}\left[1+(n-1)\rho\right]\right\} \\
&= \frac{n\tau_\varepsilon + \widehat{\tau}_s\left[1+(n-1)\rho\right]}{n\tau_\varepsilon\widehat{\tau}_s}. \qquad \text{(A19)}
\end{aligned}
$$

Using the same steps, with $\widehat{\tau}_s^{-1}$ replaced by τ_s^{-1} in (A18), we can show that

$$
var(\widehat{m}) = \frac{n\tau_\varepsilon + \tau_s\left[1+(n-1)\rho\right]}{n\tau_\varepsilon\tau_s}. \qquad \text{(A20)}
$$

This completes the proof of Lemma 3.

Proof of Proposition 3

Part (a) As shown above,

$$
\widetilde{\sigma}^2 = \underbrace{\left\{\frac{n\tau_\varepsilon}{n\tau_\varepsilon + \tau_s\left[1+(n-1)\rho\right]}\right\}^2}_{\psi^2} \cdot \underbrace{\frac{n\tau_\varepsilon + \widehat{\tau}_s\left[1+(n-1)\rho\right]}{n\tau_\varepsilon\widehat{\tau}_s}}_{var_p(\widehat{m})}.
$$

It is clear that any changes in τ_s will only affect ψ but not $var_p(\widehat{m})$. In particular, ψ (and hence $\widetilde{\sigma}^2$) is strictly decreasing in τ_s when $\rho > -1/(n-1)$. If $\rho = -1/(n-1)$, then ψ, $var_p(\widehat{m})$ and $\widetilde{\sigma}^2$ are all independent of τ_s. On the other hand, an increase in $\widehat{\tau}_s$ will lower $\widetilde{\sigma}^2$ because

$$
var_p(\widehat{m}) = \frac{1}{\widehat{\tau}_s} + \frac{\left[1+(n-1)\rho\right]}{n\tau_\varepsilon},
$$

which is strictly decreasing in $\widehat{\tau}_s$, and ψ is independent of $\widehat{\tau}_s$.

Part (b) Consider the logarithm of ψ and $var_p(\widehat{m})$,

$$
\ln\psi = \ln n + \ln\tau_\varepsilon - \ln\left\{n\tau_\varepsilon + \tau_s\left[1+(n-1)\rho\right]\right\},
$$

$$
\ln\left[var_p(\widehat{m})\right] = \ln\left\{n\tau_\varepsilon + \widehat{\tau}_s\left[1+(n-1)\rho\right]\right\} - \ln n - \ln\tau_\varepsilon - \ln\widehat{\tau}_s.
$$

Holding $\{\tau_s, \widehat{\tau}_s, \rho, n\}$ constant, consider the total derivatives of these with respect to τ_ε, i.e.,

$$\frac{d\psi}{\psi} = \frac{\tau_s [1 + (n-1)\rho]}{n\tau_\varepsilon + \tau_s [1 + (n-1)\rho]} \frac{d\tau_\varepsilon}{\tau_\varepsilon} = \frac{1 + (n-1)\rho}{n\tau_\varepsilon var(\widehat{m})} \frac{d\tau_\varepsilon}{\tau_\varepsilon}, \tag{A21}$$

$$\frac{dvar_p(\widehat{m})}{var_p(\widehat{m})} = -\frac{\widehat{\tau}_s [1 + (n-1)\rho]}{n\tau_\varepsilon + \widehat{\tau}_s [1 + (n-1)\rho]} \frac{d\tau_\varepsilon}{\tau_\varepsilon} = \frac{1 + (n-1)\rho}{n\tau_\varepsilon var_p(\widehat{m})} \frac{d\tau_\varepsilon}{\tau_\varepsilon}. \tag{A22}$$

These show that an increase in τ_ε will raise the value of ψ but lower $var_p(\widehat{m})$.

Part (c) The overall effect on $\widetilde{\sigma}^2$ is determined by

$$\frac{\tau_\varepsilon}{\widetilde{\sigma}^2} \frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} = 2\frac{\tau_\varepsilon}{\psi} \frac{d\psi}{d\tau_\varepsilon} + \frac{\tau_\varepsilon}{var_p(\widehat{m})} \frac{dvar_p(\widehat{m})}{d\tau_\varepsilon}.$$

Using (A21) and (A22), it can be shown that

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} \gtrless 0 \quad \Leftrightarrow \quad 2var_p(\widehat{m}) \gtrless var(\widehat{m}).$$

The condition on the right side is equivalent to

$$\frac{2\tau_s}{n\tau_\varepsilon + \tau_s [1 + (n-1)\rho]} \gtrless \frac{\widehat{\tau}_s}{n\tau_\varepsilon + \widehat{\tau}_s [1 + (n-1)\rho]},$$

which can be simplified to become

$$\tau_s \gtrless \frac{n\tau_\varepsilon \widehat{\tau}_s}{2n\tau_\varepsilon + \widehat{\tau}_s [1 + (n-1)\rho]}.$$

This establishes the condition in part (c).

Part (d) Holding $\{\tau_s, \widehat{\tau}_s, \tau_\varepsilon, n\}$ constant, consider the total derivatives of ψ and $var_p(\widehat{m})$ with respect to ρ, i.e.,

$$\frac{d\psi}{\psi} = -\frac{\tau_s (n-1)\rho}{n\tau_\varepsilon + \tau_s [1 + (n-1)\rho]} \frac{d\rho}{\rho},$$

$$\frac{dvar_p(\widehat{m})}{var_p(\widehat{m})} = \frac{\widehat{\tau}_s (n-1)\rho}{n\tau_\varepsilon + \widehat{\tau}_s [1 + (n-1)\rho]} \frac{d\rho}{\rho}.$$

Note that these equations are essentially the same as (A21) and (A22) but with opposite sides. The desired result can be obtained by using the same steps as in part (a). This completes the proof of Proposition 3.

Proof of Proposition 4

Part (a) Given that $Cov(s,m) = \sigma_s^2 + \rho_{s,b}\sigma_s\sigma_b$ and $var(m) = \sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2 + 2\rho_{s,b}\sigma_s\sigma_b$, we can write

$$\psi = \frac{Cov(s,m)}{var(m)} = \frac{\sigma_s^2 + \rho_{s,b}\sigma_s\sigma_b}{\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2 + 2\rho_{s,b}\sigma_s\sigma_b}. \tag{A23}$$

Straightforward differentiation gives

$$\frac{d\psi}{d\rho_{s,b}} = \frac{\sigma_s\sigma_b\left[var(m) - 2Cov(s,m)\right]}{\left[var(m)\right]^2},$$

where

$$var(m) - 2Cov(s,m) = \sigma_b^2 + \sigma_\varepsilon^2 - \sigma_s^2.$$

Hence,

$$\frac{d\widetilde{\sigma}^2}{d\rho_{s,b}} = 2\psi\frac{d\psi}{d\rho_{s,b}} \cdot var_p(\widehat{m}) \gtrless 0 \qquad \text{iff} \qquad Cov(s,m)\left[var(m) - 2Cov(s,m)\right] \gtrless 0.$$

Part (b) Differentiating the expression in (A23) with respect to τ_s gives

$$\frac{d\psi}{d\tau_s} = \frac{\left\{\left(2\sigma_s + \rho_{s,b}\sigma_b\right)var(m) - 2\sigma_s\left(\sigma_s + \rho_{s,b}\sigma_b\right)^2\right\}}{\left[var(m)\right]^2}\underbrace{\frac{d\sigma_s}{d\tau_s}}_{(-)}.$$

The expression inside the curly brackets can be simplified as follows

$$\begin{aligned}
& \left(2\sigma_s + \rho_{s,b}\sigma_b\right)var(m) - 2\sigma_s\left(\sigma_s + \rho_{s,b}\sigma_b\right)^2 \\
= & \left(2\sigma_s + \rho_{s,b}\sigma_b\right)\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2 + 2\rho_{s,b}\sigma_s\sigma_b\right) - 2\sigma_s\left(\sigma_s + \rho_{s,b}\sigma_b\right)^2 \\
= & 2\sigma_s\left(\sigma_b^2 + \sigma_\varepsilon^2\right) + \rho_{s,b}\sigma_b\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2\right).
\end{aligned}$$

Hence,

$$\frac{d\psi}{d\tau_s} = \frac{\sigma_b\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2\right)}{\left[var(m)\right]^2}\left[\rho_{s,b} + \frac{2\sigma_s\left(\sigma_b^2 + \sigma_\varepsilon^2\right)}{\sigma_b\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2\right)}\right]\underbrace{\frac{d\sigma_s}{d\tau_s}}_{(-)}.$$

This, together with

$$\psi = \frac{Cov(s,m)}{var(m)} = \frac{\sigma_s\sigma_b}{var(m)}\left(\rho_{s,b} + \frac{\sigma_s}{\sigma_b}\right),$$

implies that

$$\frac{d\widetilde{\sigma}^2}{d\tau_s} = 2\psi var_p\left(\widehat{m}\right) \cdot \frac{d\psi}{d\tau_s} \gtrless 0 \qquad \text{iff} \qquad \left(\rho_{s,b} + \frac{\sigma_s}{\sigma_b}\right)\left[\rho_{s,b} + \frac{2\left(\sigma_b^2 + \sigma_\varepsilon^2\right)}{\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2\right)} \frac{\sigma_s}{\sigma_b}\right] \lessgtr 0.$$

Part (c) Differentiating the expression in (A23) with respect to τ_b gives

$$\frac{d\psi}{d\tau_b} = \frac{\sigma_s\left\{\rho_{s,b} var\left(m\right) - 2\left(\sigma_s + \rho_{s,b}\sigma_b\right)\left(\sigma_b + \rho_{s,b}\sigma_s\right)\right\}}{\left[var\left(m\right)\right]^2} \underbrace{\frac{d\sigma_b}{d\tau_b}}_{(-)}.$$

The term inside the curly brackets can be simplified as follows:

$$\begin{aligned} & \rho_{s,b} var\left(m\right) - 2\left(\sigma_s + \rho_{s,b}\sigma_b\right)\left(\sigma_b + \rho_{s,b}\sigma_s\right) \\ = \; & \rho_{s,b}\left(\sigma_s^2 + \sigma_b^2 + \sigma_\varepsilon^2 + 2\rho_{s,b}\sigma_s\sigma_b\right) - 2\left(\sigma_s + \rho_{s,b}\sigma_b\right)\left(\sigma_b + \rho_{s,b}\sigma_s\right) \\ = \; & -\left[\rho_{s,b}\left(\sigma_s^2 + \sigma_b^2 - \sigma_\varepsilon^2\right) + 2\sigma_s\sigma_b\right]. \end{aligned}$$

Hence,

$$\frac{d\psi}{d\tau_b} = -\frac{-\sigma_s}{\left[var\left(m\right)\right]^2}\left[\rho_{s,b}\left(\sigma_s^2 + \sigma_b^2 - \sigma_\varepsilon^2\right) + 2\sigma_s\sigma_b\right] \underbrace{\frac{d\sigma_b}{d\tau_b}}_{(-)}$$

It follows that

$$\frac{d\widetilde{\sigma}^2}{d\tau_b} = 2\psi var_p\left(\widehat{m}\right) \cdot \frac{d\psi}{d\tau_b} \gtrless 0 \qquad \text{iff} \qquad \left(\rho_{s,b} + \frac{\sigma_s}{\sigma_b}\right)\left[\rho_{s,b}\left(\sigma_s^2 + \sigma_b^2 - \sigma_\varepsilon^2\right) + 2\sigma_s\sigma_b\right] \gtrless 0.$$

Part (d) This result follows immediately from the fact that ψ is independent of $\left\{\widehat{\tau}_s, \widehat{\tau}_b, \widehat{\rho}_{s,b}\right\}$.

Part (e) Since $Cov\left(s,m\right)$ is independent of τ_ε, straightforward differentiation yields

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} = \left[Cov\left(s,m\right)\right]^2\left\{\frac{1}{\left[var\left(m\right)\right]^2}\frac{dvar_p\left(m\right)}{d\tau_\varepsilon} - 2\frac{var_p\left(m\right)}{\left[var\left(m\right)\right]^3}\frac{dvar_p\left(m\right)}{d\tau_\varepsilon}\right\}.$$

Note that

$$\frac{dvar\left(m\right)}{d\tau_\varepsilon} = \frac{dvar_p\left(m\right)}{d\tau_\varepsilon} = \frac{d\sigma_\varepsilon^2}{d\tau_\varepsilon} = -\sigma_\varepsilon^4 < 0.$$

Combining these two equations gives

$$\frac{d\widetilde{\sigma}^2}{d\tau_\varepsilon} = \left[\frac{Cov\left(s,m\right)}{var\left(m\right)}\right]^2\left[\frac{2var_p\left(m\right)}{var\left(m\right)} - 1\right]\sigma_\varepsilon^4 \gtrless 0 \qquad \text{iff} \qquad 2var_p\left(m\right) \gtrless var\left(m\right).$$

This completes the proof of Proposition 4.

Derivation of Equation (32) in the Paper

Consider an arbitrary voter with $\delta_v \in \mathbb{R}$. Conditional on $\mathbf{m}$, the voter's expected utility if R wins is

$$\begin{aligned} & E\left[U\left(x^*_{eq};\delta_v\right) \mid \mathbf{m}\right] \\ = & -E\left[\left(\delta_v + \psi\widehat{m} - x^*_{eq}\right)^2 \mid \mathbf{m}\right] - var\left(s \mid \mathbf{m}\right) \\ = & -\left\{\left(\delta_v - x^*_{eq}\right)^2 + 2\left(\delta_v - x^*_{eq}\right)\psi\widehat{m} + \left(\psi\widehat{m}\right)^2 + var\left(s \mid \mathbf{m}\right)\right\}. \end{aligned}$$

Similarly, the voter's expected utility if L wins is

$$E\left[U\left(-x^*_{eq};\delta_v\right) \mid \mathbf{m}\right] = -\left\{\left(\delta_v + x^*_{eq}\right)^2 + 2\left(\delta_v + x^*_{eq}\right)\psi\widehat{m} + \left(\psi\widehat{m}\right)^2 + var\left(s \mid \mathbf{m}\right)\right\}.$$

Before $\mathbf{m}$ is realised, the voter's expected utility is

$$\begin{aligned} E\left[U\left(x^*_{eq};\delta_v\right)\right] &= \int_0^\infty E\left[U\left(x^*_{eq};\delta_v\right) \mid \mathbf{m}\right] dG\left(\widehat{m}\right) + \int_{-\infty}^0 E\left[U\left(-x^*_{eq};\delta_v\right) \mid \mathbf{m}\right] dG\left(\widehat{m}\right) \\ &= -\left[\frac{1}{2}\left(\delta_v - x^*_{eq}\right)^2 + \frac{1}{2}\left(\delta_v + x^*_{eq}\right)^2 + var\left(s \mid \mathbf{m}\right)\right] \\ &\quad -\left[2\left(\delta_v - x^*_{eq}\right)\psi\int_0^\infty \widehat{m}dG\left(\widehat{m}\right) + 2\left(\delta_v + x^*_{eq}\right)\psi\widehat{m}\int_{-\infty}^0 \widehat{m}dG\left(\widehat{m}\right)\right] \\ &\quad -\psi^2\int_{-\infty}^\infty \widehat{m}^2 dG\left(\widehat{m}\right) \\ &= 4\psi x^*_{eq}\int_0^\infty \widehat{m}dG\left(\widehat{m}\right) - \left[\delta_v^2 + \left(x^*_{eq}\right)^2 + var\left(s \mid \mathbf{m}\right) + \psi^2 var\left(\widehat{m}\right)\right]. \qquad \text{(A24)} \end{aligned}$$

The last line follows from the fact that $G\left(\cdot\right)$ is the CDF of a symmetric distribution around zero, hence

$$\int_{-\infty}^\infty \widehat{m}dG\left(\widehat{m}\right) = 0 \qquad \text{and} \qquad \int_{-\infty}^0 \widehat{m}dG\left(\widehat{m}\right) = -\int_0^\infty \widehat{m}dG\left(\widehat{m}\right).$$

Using the formula,

$$\int_0^\infty x^{2n+1}\exp\left(-Ax^2\right)dx = \frac{n!}{2A^{n+1}}, \qquad \text{for } A > 0 \text{ and } n = 0, 1, 2, ...,$$

we can get

$$\int_0^\infty \widehat{m} dG(\widehat{m}) = \frac{1}{\sqrt{2\pi}\sigma_m}\int_0^\infty \widehat{m}\exp\left[-\left(2\sigma_m^2\right)^{-1}(\widehat{m})^2\right]d\widehat{m} = \frac{\sigma_m}{\sqrt{2\pi}}.$$

Substituting this into (A24) gives

$$E\left[U\left(x_{eq}^*;\delta_v\right)\right] = 2\sqrt{\frac{2}{\pi}}x_{eq}^*\psi\sigma_m - \left(x_{eq}^*\right)^2 - \left[\delta_v^2 + var\left(s \mid \mathbf{m}\right) + \psi^2 var\left(\widehat{m}\right)\right]. \tag{A25}$$

Finally, by the law of total variance, we can get

$$var\left(s \mid \mathbf{m}\right) + \psi^2 var\left(\widehat{m}\right) = \tau_s^{-1}. \tag{A26}$$

To see this, first recall that μ_s in the voter's subjective prior belief is normalised to zero, hence the variance of s in their prior belief is given by

$$\tau_s^{-1} = var\left(s\right) = E\left(s^2\right) = E\left[E\left(s^2 \mid \mathbf{m}\right)\right],$$

where the outer expectation is taken with respect to the joint distribution of $\mathbf{m}$. It follows that

$$\begin{aligned}\tau_s^{-1} &= E\left\{var\left(s \mid \mathbf{m}\right) + \left[E\left(s \mid \mathbf{m}\right)\right]^2\right\} \\ &= var\left(s \mid \mathbf{m}\right) + \int_{-\infty}^{\infty}\left[E\left(s \mid \mathbf{m}\right)\right]^2 dG\left(\widehat{m}\right). \end{aligned} \tag{A27}$$

The last line uses the facts that $var\left(s \mid \mathbf{m}\right)$ is a deterministic constant according to Equation (3) in the paper and $E\left(s \mid \mathbf{m}\right)$ is a function of $\widehat{m}$. Since the expected value of $\widehat{m}$ is zero,

$$\int_{-\infty}^{\infty}\left[E\left(s \mid \mathbf{m}\right)\right]^2 dG\left(\widehat{m}\right) = var\left[E\left(s \mid \mathbf{m}\right)\right] = \psi^2 var\left(\widehat{m}\right). \tag{A28}$$

Substituting (A28) into (A27) gives (A26). Combining (A25) and (A26) gives

$$E\left[U\left(x_{eq}^*;\delta_v\right)\right] = 2\sqrt{\frac{2}{\pi}}x_{eq}^*\psi\sigma_m - \left(x_{eq}^*\right)^2 - \left(\delta_v^2 + \tau_s^{-1}\right),$$

which is Equation (32) in the paper.

Proof of Proposition 5

The second inequality in Equation (33) in the paper can be rewritten as

$$2\sqrt{\frac{2}{\pi}}\psi\sigma_m \geq x_{eq}^* = \frac{2\phi - \gamma h(0)}{4h(0)\phi + 2}$$

$$\Leftrightarrow 2\sqrt{2}\psi\sigma_m\left[4h(0)\phi + 2\right] \geq 2\sqrt{\pi}\phi - \sqrt{\pi}\gamma h(0)$$

$$\Leftrightarrow 4\sqrt{2}\psi\sigma_m + \sqrt{\pi}\gamma h(0) \geq 2\left[\sqrt{\pi} - 4\sqrt{2}\psi\sigma_m h(0)\right]\phi.$$

There are two possible cases: If $\sqrt{\pi} - 4\sqrt{2}\psi\sigma_m h(0) \leq 0$, or equivalently,

$$\frac{\widetilde{\sigma}}{\psi\sigma_m} = \sqrt{\frac{var_p(\widehat{m})}{var(\widehat{m})}} \leq \frac{4}{\pi},$$

then the second inequality in Equation (33) is automatically satisfied. This means $E\left[U\left(x_{eq}^*; \delta_v\right)\right] \geq E\left[U(0; \delta_v)\right]$ for any $x_{eq}^* \geq 0$.

But if $\sqrt{\pi} - 4\sqrt{2}\psi\sigma_m h(0) > 0$, or equivalently,

$$\frac{\widetilde{\sigma}}{\psi\sigma_m} = \sqrt{\frac{var_p(\widehat{m})}{var(\widehat{m})}} > \frac{4}{\pi},$$

then the second inequality in Equation (33) holds if and only if

$$\phi \leq \frac{\sqrt{2}\psi\sigma_m + \sqrt{\pi}\gamma h(0)}{2\left[\sqrt{\pi} - 4\sqrt{2}\psi\sigma_m h(0)\right]} = \frac{\sqrt{\pi}\left(8\psi\sigma_m \cdot \widetilde{\sigma} + \gamma\right)}{2\sqrt{2}\left[\pi\widetilde{\sigma} - 4\psi\sigma_m\right]}.$$

This completes the proof of Proposition 5.

Proof of Proposition 6

Suppose $var(\widehat{m}) = var_p(\widehat{m})$, which implies $\psi\sigma_m = \widetilde{\sigma}$. Then Equation (32) in the paper can be rewritten as

$$E\left[U\left(x_{eq}^*; \delta_v\right)\right] = 2\sqrt{\frac{2}{\pi}}\widetilde{\sigma}x_{eq}^* - \left(x_{eq}^*\right)^2 - \left(\delta_v^2 + \tau_s^{-1}\right),$$

for any $x_{eq}^* > 0$, or equivalently $\widetilde{\sigma} > \sigma_{\min}$. Straightforward differentiation yields

$$\begin{aligned}\frac{d}{dz}E\left[U\left(x_{eq}^*; \delta_v\right)\right] &= 2\sqrt{\frac{2}{\pi}}\left(\widetilde{\sigma}\frac{dx_{eq}^*}{dz} + x_{eq}^*\frac{d\widetilde{\sigma}}{dz}\right) - 2x_{eq}^* \cdot \frac{dx_{eq}^*}{dz} \\ &= \left[2\left(\sqrt{\frac{2}{\pi}}\widetilde{\sigma} - x_{eq}^*\right)\frac{dx_{eq}^*}{d\widetilde{\sigma}} + 2\sqrt{\frac{2}{\pi}}x_{eq}^*\right]\frac{d\widetilde{\sigma}}{dz}.\end{aligned}$$

The second line uses the chain rule of differentiation,

$$\frac{dx^*_{eq}}{dz} = \frac{dx^*_{eq}}{d\widetilde{\sigma}} \cdot \frac{d\widetilde{\sigma}}{dz}.$$

As shown in Corollary 1, x^*_{eq} is strictly increasing in $\widetilde{\sigma}$ whenever $\widetilde{\sigma} > \sigma_{\min}$. This, together with $x^*_{eq} > 0$ and condition (35) in the paper, means that $\sqrt{2/\pi}\widetilde{\sigma} \geq x^*_{eq}$ is a sufficient condition for

$$\frac{d}{dz} E\left[U\left(x^*_{eq}; \delta_v\right)\right] > 0.$$

Recall that the extent of polarisation x^*_{eq} is determined by

$$x^*_{eq} = \frac{2\phi - \gamma h(0)}{4h(0)\phi + 2}, \qquad \text{where } h(0) \equiv 1/\left(\widetilde{\sigma}\sqrt{2\pi}\right).$$

Hence, x^*_{eq} can also be expressed as

$$x^*_{eq} = \frac{2\sqrt{2\pi}\phi\widetilde{\sigma} - \gamma}{4\phi + 2\sqrt{2\pi}\widetilde{\sigma}} = \frac{2\sqrt{2\pi}\phi\left(\widetilde{\sigma} - \sigma_{\min}\right)}{4\phi + 2\sqrt{2\pi}\widetilde{\sigma}}, \qquad \text{where } \sigma_{\min} \equiv \frac{\gamma}{2\sqrt{2\pi}\phi}.$$

The sufficient condition $\sqrt{2/\pi}\widetilde{\sigma} \geq x^*_{eq}$ can now be rewritten as

$$\widetilde{\sigma} \geq \frac{\pi\phi\left(\widetilde{\sigma} - \sigma_{\min}\right)}{2\phi + \sqrt{2\pi}\widetilde{\sigma}}$$

$$\Leftrightarrow \widetilde{\sigma}\left(2\phi + \sqrt{2\pi}\widetilde{\sigma}\right) \geq \pi\phi\left(\widetilde{\sigma} - \sigma_{\min}\right)$$

$$\Leftrightarrow \sqrt{2\pi}\widetilde{\sigma}^2 - \phi(\pi - 2)\widetilde{\sigma} + \pi\phi\sigma_{\min} \geq 0. \tag{A29}$$

Consider the following quadratic equation:

$$\sqrt{2\pi}y^2 - \phi(\pi - 2)y + \pi\phi\sigma_{\min} = 0.$$

Since $\phi(\pi - 2) > 0$ and $\pi\phi\sigma_{\min} > 0$, this equation has two distinct real roots. The sum and the product of roots are, respectively, given by $-\phi(\pi - 2) < 0$ and $\pi\phi\sigma_{\min} > 0$. Hence, the two roots must be negative-valued. This in turn implies that $\sqrt{2\pi}y^2 - \phi(\pi - 2)y + \pi\phi\sigma_{\min} > 0$ for all $y \geq 0$. Hence, (A29) is valid for any $\widetilde{\sigma} > \sigma_{\min} > 0$. This proves the desired result.

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